CONTROLLABILITY, OBSERVABILITY AND DUALITY IN LINEAR SYSTEMS WITH MULTIPLE NORM-BOUNDED CONTROLLERS

By

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a

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The undersigned certify that we have read a thesis, entitled "Controllabilities, Observability and Duality in Linear Systems with Multiple Norm-bounded Controllers" submitted to the Graduate School by Mr LI Chun Wah (李鎮華) in partial fulfillment of the requirements for the degree of Master of Philosophy in Mathematics. We recommend that it be accepted.

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Preface

Over the past few years, linear control systems with input constraints play an important role in system theory. The purpose of this thesis is to study their controllability, observability and duality. Necessary and sufficient conditions for such systems to be controllable will be given. The thesis is divided into two main chapters. The first one concerns with the finite dimensional case while the second one generalizes most of the results to the infinite dimensional situation. Pertinent references are given at the end of each chapter.

Chapter 1

Finite Dimensional Linear Systems With Multiple Norm-bounded Controllers

I. Introduction

Recently there is much interest in the study of constrained controllability in linear systems. A recent report on the subject is given by Barmish and Schmitendorf [1]. When the constraint for the controls is a ball, Antosiewicz [2] appears to be one of the early investigator giving a systematic account (see also Marzollo [3]). Subsequently Conti [4] and Pandolfi [5] made substantial contributions and advanced the knowledge in the same direction. The present chapter aims at the study of controllability of linear systems with multiple independent controllers, each of which is norm bounded or confined to a ball. Systems with many controllers receive much attention in two major fronts, namely, in decentralized control (cooperative mode) and in differential game (non-cooperative mode). For unconstrained controllers, these problems have been studied by many [6] [7] [8] [9]. However, there appears not much work in the constrained case.

Our study naturally includes both the usual (cooperative) controllability as well as the max-min (non-cooperative) controllability [7]. We also consider the situation where there are two opposing teams. Very often, the final state of the system is required to be sent not only to the origin but to some closed and convex target set. We give an account for this problem with emphasis on some important special cases such as a ball and a subspace. For the cooperative mode we give a duality theory between controllability and observability, extending Kalman's fundamental result to the constrained case. Finally we consider discrete time systems giving a parallel development analogous to the continuous time case. For brevity, we omit most of the proofs but also point out some distinct features.

II. The basic problems and preliminaries

Consider the multi-inputs control system described by the linear differential equations, with μ independent controllers

$$\dot{x} = A(t) + \sum_{i=1}^{\mu} B_i(t)u_i(t) \qquad (2.1)$$
$$x(t_o) = x_o$$

where x is an n dimensional state vector; $A(t)$ is an $n \times n$ matrix with components being measurable on the fixed time interval $T = [t_o, t_1]$; $B_i(t)$ is an $n \times h_i$ matrix with components being q_i power Lebesgue integrable on T while u_i is an h_i dimensional input vector belongs to $L_{r_i,p_i}(T)$ vector value Lebesgue integrable space with $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = 1, \ldots, \mu$.

Define the norms

$$\|u_i(t)\|_{r_i} = \left(\sum_{j=1}^{h_i} |u_i^j(t)|^{r_i} \right)^{1/r_i} \quad \text{if} \quad 1 \leq r_i < \infty \qquad (2.2)$$
$$\|u_i(t)\|_{\infty} = \max_{j=1,\ldots,h_i} \{|u_i^j(t)|\}$$

and

$$|||u_i(t)||_{r_i}|_{p_i} = \left(\int_{t_o}^{t_1} \|u_i(t)\|_{r_i}^{p_i} \, dt \right)^{1/p_i} \quad \text{if} \quad 1 \leq p_i < \infty \qquad (2.3)$$
$$|||u_i(t)||_{r_i}|_{\infty} = \operatorname*{ess\,sup}_{t \in T} \{\|u_i(t)\|_{r_i}\} .$$

We consider the input constraints to be

$$U_i^{\rho_i} = \{u_i \in L_{r_i,p_i}(T) : |||u_i||_{r_i}|_{p_i} \leq \rho_i\} , \quad \rho_i > 0 \qquad (2.4)$$
$$i = 1, \ldots, \mu .$$

Let the state transition matrix of (2.1) be $\Phi(t, \tau)$. The variation of constant formula is hence

$$x(t) = \Phi(t, t_o)x_o + \sum_{i=1}^{\mu} \int_{t_o}^{t} \Phi(t, \tau)B_i(\tau)u_i(\tau)d\tau . \tag{2.5}$$

Denote

$$\Lambda_i(u_i) = \int_{t_o}^{t_1} \Phi(t_1, \tau)B_i(\tau)u_i(\tau)d\tau \tag{2.6}$$

$$R_i = \{\Lambda_i(u_i), u_i \in U_i^{\rho_i}\} \tag{2.7}$$

$i = 1, \ldots, \mu$.

Lemma 2.1. The sets $U_i^{\rho_i}$, $i = 1, \ldots, \mu$ are convex and weakly compact in $L_{r_i,p_i}(T)$, $i = 1, \ldots, \mu$ respectively, where $1 < p_i \leqslant \infty$. When $p_i = \infty$, $U_i^{\rho_i}$ is convex and weakly compact in $L_{r_i,2}(T)$.

Proof: From (2.4), we see that $U_i^{p_i}$, $i = 1, \ldots, \mu$ are closed, bounded and convex. Thus, they are weakly closed. If $1 < p_i < \infty$, the spaces $L_{r_i,p_i}(T)$, $i = 1, \ldots, \mu$ are reflexive Banach spaces and hence, bounded weakly closed sets in $L_{r_i,p_i}(T)$ are weakly compact. If $p_i = \infty$, then $\|u_i(t)\|_{r_i} \leqslant \rho_i$ a.e. $t \in T$. Thus,

$$\left(\int_{t_o}^{t_1} \|u_i(t)\|_{r_i}^2 dt\right)^{\frac{1}{2}} \leqslant \rho_i(t_1 - t_o)^{\frac{1}{2}} .$$

Let $M = \rho_i(t_1 - t_o)^{\frac{1}{2}}$, then $u_i \in L_{r_i,2}(T)$. Hence, $U_i^{\rho_i}$ is a weakly closed subset of a weakly compact set

$$\{w \in L_{r_i,2}(T) : |\|w\|_{r_i}|_2 \leqslant M\}$$

and so $U_i^{\rho_i}$ is weakly compact, in $L_{r_i,2}(T)$.

Lemma 2.2. The mappings

$$\Lambda_i : L_{r_i,p_i}(T) \to \mathbb{R}^n , \qquad i = 1, \ldots, \mu ; \quad 1 < p_i < \infty$$

defined by (2.6) are continuous linear compact operators. Thus, R_i defined by (2.7) are compact and convex.

Proof: Linearity of Λ_i is obvious and

$$\|\Lambda_i(u_i)\| \leq \int_{t_o}^{t_1} \|\Phi(t_1, \tau)B_i(\tau)u_i(\tau)\| d\tau$$
$$\leq \left(\int_{t_o}^{t_1} \|\Phi(t_1, \tau)B_i(\tau)\|_{s_i}^{q_i} d\tau\right)^{1/q_i} \left(\int_{t_o}^{t_1} \|u_i(\tau)\|_{r_i}^{p_i} d\tau\right)^{1/p_i}$$

where $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $\frac{1}{r_i} + \frac{1}{s_i} = 1$, $i = 1, \ldots, \mu$ and the matrix norm defined by

$$\|\Phi(t_1, \tau)B_i(\tau)\|_{s_i} = \sup\{\|\Phi(t_1, \tau)B_i(\tau)y\|_{s_i} , \|y\|_{s_i} = 1\} .$$

Thus, Λ_i is continuous and hence weakly continuous. For each i , it maps weakly compact balls $U_i^{\rho_i}$ to some weakly compact set in $\mathbb{R}^n$ where weak and strong compactness coincide. Hence, Λ_i $i = 1, \ldots, \mu$ are compact and $R_i = \Lambda_i(U_i^{\rho_i})$ are convex and compact. If $p_i = \infty$, then $U_i^{\rho_i}$ is weakly compact in $L_{r_i,2}(T)$. Thus, $R_i = \Lambda_i(U_i^{\rho_i})$ is compact in $\mathbb{R}^n$ by continuity of $\Lambda_i : L_{r_i,2}(T) \to \mathbb{R}^n$.

III. Main Results on Controllability

We consider the control problem of determining whether or not there exist imputs $u_i \in U_i^{\rho_i}$, $i = 1, \ldots, \mu$ which steer the initial state $x_o \neq 0$ to the origin at t_1, i.e. $x(t_1) = 0$.

Definition 3.1. The system (2.1) is said to be null-controllable on T if there exist inputs $u_i \in U_i^{\rho_i}$, $i = 1, \ldots, \mu$ which can steer the initial state $x_o \neq 0$ to the origin.

In order to establish the main theorem, we need the following lemmas.

Lemma 3.1. If X and Y are closed and convex sets of $\mathbb{R}^n$ with one of them being compact, then

(i) $X \cap Y \neq \phi$ iff for each $g \in \mathbb{R}^n$, we have

$$\inf_{x \in X} \langle g, x \rangle \leq \sup_{y \in Y} \langle g, y \rangle . \tag{3.1}$$

(ii) $X \subset Y$ iff for each $g \in \mathbb{R}^n$, we have

$$\sup_{x \in X} \langle g, x \rangle \leq \sup_{y \in Y} \langle g, y \rangle . \tag{3.2}$$

Proof: (i) If $w \in X \cap Y$ then for all $g \in \mathbb{R}^n$

$$\inf_{x \in X} \langle g, x \rangle \leq \langle g, w \rangle \leq \sup_{y \in Y} \langle g, y \rangle .$$

Conversely, if $X \cap Y = \phi$, then by the strict separation theorem, there exists a non-zero vector $\bar{g}$ such that

$$\inf_{x \in X} \langle \bar{g}, x \rangle > \sup_{y \in Y} \langle \bar{g}, y \rangle$$

which contradicts (3.1).

(ii) If $X \subset Y$ then (3.2) holds for all $g \in \mathbb{R}^n$. Conversely, if $\exists\, z \in X \setminus Y$, then by the strict separation theorem, there exists a non-zero vector $\hat{g}$ such that

$$\sup_{y \in Y} \langle \hat{g}, y \rangle < \langle \hat{g}, z \rangle \leq \sup_{x \in X} \langle \hat{g}, x \rangle$$

Thus, it contradicts (3.2).

Lemma 3.2. If X and Y are closed convex sets of $\mathbb{R}^n$ with

X being compact, then $X + Y$ is closed and convex. In addition, if Y is also compact, then $X + Y$ is compact.

Proof: Convexity of $X + Y$ is obvious. Let $z \in \overline{X + Y}$, then $\exists \{x_n\} \subset X$, $\{y_n\} \subset Y$ such that $x_n + y_n \to z$. Since X is compact, we have a convergent subsequence $x_{n_k} \to x \in X$. Thus, $\exists \{y_{n_i}\} \subset \{y_{n_k}\}$ such that $y_{n_i} \to y \in Y$. Hence $x_{n_i} + y_{n_i} \to x + y$ and so $z = x + y \in X + Y$. Therefore $X + Y$ is closed. If X and Y are compact, then they are bounded and so is $X + Y$. Consequently, $X + Y$ is compact.

Now, we are in the position to establish the main theorem as follows.

Theorem 3.1. The system (2.1) is null-controllable on T iff for each $g \in \mathbb{R}^n$, we have

$$|\langle g, \Phi(t_1, t_o)x_o\rangle| \leqslant \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_i^*(t)\, \Phi^*(t_1, \tau)g\|_{s_i}^{q_i} \, d\tau \right)^{1/q_i} \tag{3.3}$$

where $\frac{1}{r_i} + \frac{1}{s_i} = 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $1 \leqslant r_i \leqslant \infty$, $1 < p_i \leqslant \infty$, $i = 1, \ldots, \mu$.

Proof: From lemma 2.2, we see that R_i, $i = 1, \ldots, \mu$, are convex and compact and so is $\sum_{i=1}^{\mu} R_i$ by lemma 3.2. In view of definition 3.1, the system (2.1) is null-controllable iff

$$-\Phi(t_1, t_o)x_o \in \sum_{i=1}^{\mu} R_i ,$$

or iff for each $g \in \mathbb{R}^n$ we have.

$$\langle g_1 - \Phi(t_1, t_o)x_o\rangle \leqslant \sup_{\substack{u_i \in U_i^{\rho_i} \\ i=1,\ldots,\mu}} \langle g, \sum_{i=1}^{\mu} \int_{t_o}^{t_1} \Phi(t_1, \tau)B_i(\tau)u_i(\tau)d\tau \rangle$$

$$= \sum_{i=1}^{\mu} \sup_{u_i \in U_i^{\rho_i}} \langle g, \int_{t_o}^{t_1} \Phi(t_1, \tau) B_i(\tau) u_i(\tau) d\tau \rangle$$

$$= \sum_{i=1}^{\mu} \sup_{u_i \in U_i^{\rho_i}} \int_{t_o}^{t_1} \langle B_i^*(\tau) \Phi^*(t_1, \tau) g, u_i(\tau) \rangle d\tau$$

$$= \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \| B_i^*(\tau) \Phi^*(t_1, \tau) g \|_{s_i}^{q_i} d\tau \right)^{1/q_i} \tag{3.4}$$

by lemma 3.1(i). Replacing g by $-g$ in (3.4), we thus have (3.3).

Remark 3.1. In view of (3.3), $g = 0$ is trivially satisfied. If $g \neq 0$, then consider $g/\|g\|_2$ with unit norm and (3.3) also holds. Hence, the system (2.1) is null-controllable on T iff (3.3) holds for all $g \in \mathbb{R}^n$ with $\|g\|_2 = 1$.

IV. Game problem

In this section, we consider the non-cooperative game problem with two teams. One team represents the pursuer while another represents the evader and the game is governed by the following linear differential equation

$$\dot{x} = A(t)x + \sum_{i=1}^{\mu} B_p^i(t) u_i(t) + \sum_{j=1}^{\nu} B_e^j(t) v_j(t) \tag{4.1}$$

$$x(t_o) = x_o \neq 0 .$$

As before, the control inputs are restricted in some constrained sets

$$U_i^{\rho_i} = \{ u_i \in L_{r_i, p_i}(T) : \| \| u_i \|_{r_i} \|_{p_i} \leq \rho_i \} \quad i = 1, \ldots, \mu , \tag{4.2}$$

$$V_j^{\sigma_j} = \{ v_j \in L_{r_j', p_j'}(T) : \| \| v_j \|_{r_j'} \|_{p_j'} \leq \sigma_j \} \quad j = 1, \ldots, \nu , \tag{4.3}$$

where $\frac{1}{r'_j} + \frac{1}{s'_j} = 1$, $\frac{1}{p'_j} + \frac{1}{q'_j} = 1$, $1 \leq r'_j \leq \infty$, $1 < p'_j \leq \infty$. Define

$$\Lambda^i_p(u_i) = \int_{t_o}^{t_1} \Phi(t_1, \tau) B^i_p(\tau) u_i(\tau) d\tau \tag{4.4}$$

$$\Lambda^j_e(v_j) = \int_{t_o}^{t_1} \Phi(t_1, \tau) B^j_e(\tau) v_j(\tau) d\tau \tag{4.5}$$

$$R^i_p = \{\Lambda^i_p(u_i), \ u_i \in U_i^{\rho_i}\} \tag{4.6}$$

$$R^j_e = \{\Lambda^j_e(v_j), \ v_j \in V_j^{\sigma_j}\} \tag{4.7}$$

$$i = 1, \ldots, \mu; \quad j = 1, \ldots, \nu .$$

In view of lemma 2.1 and lemma 2.2, we see that $U_i^{\rho_i}$, $V_j^{\sigma_j}$ are convex and weakly compact, Λ^i_p , Λ^j_e are continuous linear compact operators and R^i_p , R^j_e are convex compact sets.

Definition 4.1. The game system (4.1) is said to be max-min null-controllable on T if for each announced evaders' controls $v_j \in V_j^{\sigma_j}$, $j = 1, \ldots, \nu$, there exist pursuers' controls $u_i \in U_i^{\rho_i}$, $i = 1, \ldots, \mu$ which steer the initial state x_o to the origin at t_1 in the sense

$$\begin{aligned} x(t_1) = \Phi(t_1, t_o)x_o &+ \sum_{i=1}^{\mu} \int_{t_o}^{t_1} \Phi(t_1, \tau) B^i_p(\tau) u_i(\tau) d\tau \\ &+ \sum_{j=1}^{\nu} \int_{t_o}^{t_1} \Phi(t_1, \tau) B^j_e(\tau) v_j(\tau) d\tau \\ = 0 . \end{aligned} \tag{4.8}$$

Analogous to theorem 3.1, we establish the following.

Theorem 4.1. The game system (4.1) is max-min null-controllable on T iff for all $g \in \mathbb{R}^n$, we have

$$|\langle g, \Phi(t_1, t_o)x_o\rangle| = \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_p^{i*}(\tau)\Phi^*(t_1, \tau)g\|_{s_i}^{q_i} d\tau\right)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \left(\int_{t_o}^{t_1} \|B_e^{j*}(\tau)\Phi^*(t_1, \tau)g\|_{s'_j}^{q'_j} d\tau\right)^{1/q'_j} \qquad (4.9)$$

Proof: Since R_p^i, R_e^j, $i = 1, \ldots, \mu$; $j = 1, \ldots, \nu$, are convex and compact and by lemma 3.2 so are $\Phi(t_1, t_o)x_o + \sum_{j=1}^{\nu} R_e^j$ and $\sum_{i=1}^{\mu} R_p^i$. In view of definition 4.1, the game system (4.1) is max-min null-controllable on T iff

$$\Phi(t_1, t_o)x_o + \sum_{j=1}^{\nu} R_e^j \subseteq -\sum_{i=1}^{\mu} R_p^i = \sum_{i=1}^{\mu} R_p^i \qquad (4.10)$$

By lemma 3.1(ii), (4.10) holds iff for all $g \in \mathbb{R}^n$, so that

$$\sup_{\substack{v_j \in V_j^{\sigma_j} \\ j=1,\ldots,\nu}} \langle g, \Phi(t_1, t_o)x_o + \sum_{j=1}^{\nu} \Lambda_e^j(v_j)\rangle \leq \sup_{\substack{u_i \in U_i^{\rho_i} \\ i=1,\ldots,\mu}} \langle g, \sum_{i=1}^{\mu} \Lambda_p^i(u_i)\rangle . \qquad (4.11)$$

The left hand side of (4.11) becomes

$$\langle g, \Phi(t_1, t_o)x_o\rangle + \sum_{j=1}^{\nu} \sup_{v_j \in V_j^{\sigma_j}} \langle g, \int_{t_o}^{t_1} \Phi(t_1, \tau)B_e^j(\tau)v_j(\tau)d\tau\rangle$$
$$= \langle g, \Phi(t_1, t_o)x_o\rangle + \sum_{j=1}^{\nu} \sup_{v_j \in V_j^{\sigma_j}} \int_{t_o}^{t_1} \langle B_e^{j*}(\tau)\, \Phi^*(t_1, \tau)g,\ v_j(\tau)\rangle d\tau$$
$$= \langle g, \Phi(t_1, t_o)x_o\rangle + \sum_{j=1}^{\nu} \sigma_j\left(\int_{t_o}^{t_1} \|B_e^{j*}(\tau)\, \Phi^*(t_1, \tau)g\|_{s'_j}^{q'_j} d\tau\right)^{1/q'_j} .$$

The right hand side of (4.11) becomes

$$\sup_{\substack{u_i \in U_i^{\rho_i} \\ i=1,\ldots,\mu}} \langle g, \sum_{i=1}^{\mu} \int_{t_o}^{t_1} \Phi(t_1, \tau) B_p^i(\tau) u_i(\tau) d\tau \rangle$$

$$= \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \| B_p^{i*}(\tau) \, \Phi^*(t_1, \tau) g \|_{s_i}^{q_i} d\tau \right)^{1/q_i} .$$

Thus,

$$\langle g, \Phi(t_1, t_o) x_o \rangle \leq \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \| B_p^{i*}(\tau) \, \Phi^*(t_1, \tau) g \|_{s_i}^{q_i} d\tau \right)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \left(\int_{t_o}^{t_1} \| B_e^{j*}(\tau) \, \Phi^*(t_1, \tau) g \|_{s_j'}^{q_j'} d\tau \right)^{1/q_j'} . \qquad (4.12)$$

Replacing g by $-g$ in (4.12), we obtain (4.9).

Remark 4.1. As discussed in remark 3.1, the game system (4.1) is max-min null-controllable on T iff (4.9) holds for all $g \in \mathbb{R}^n$ with $\|g\|_2 = 1$.

Remark 4.2. If $\nu = 0$, i.e. there are no evaders' controls then theorem 4.1 reduces to theorem 3.1.

V. <u>Controllability to some target set</u>

We have considered the null-controllability in the previous sections. However, very often, we only need the final state $x(t_1)$ be sent to some target set Ω in $\mathbb{R}^n$.

Definition 5.1. The system (4.1) is said to be max-min Ω-controllable on T if for each announced $v_j \in V_j^{\sigma_j}$, $j = 1, \ldots, \nu$, there exist $u_i \in U_i^{\rho_i}$, $i = 1, \ldots, \mu$ which steer the initial state x_o to Ω at t_1, i.e. $x(t_1) \in \Omega$.

Theorem 5.1. If the target set Ω is closed and convex, then the system (4.1) is max-min Ω-controllable on T iff for all $g \in \mathbb{R}^n$, we have

$$\langle g, \Phi(t_1, t_o)x_o\rangle \leq \sup_{y\in\Omega} \langle g, y\rangle + \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_p^{i*}(\tau)\, \Phi^*(t_1, \tau)g\|_{s_i}^{q_i} d\tau\right)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \left(\int_{t_o}^{t_1} \|B_e^{j*}(\tau)\, \Phi^*(t_1, \tau)g\|_{s_i'}^{q_i'} d\tau\right)^{1/q_i'} . \tag{5.1}$$

Proof: Since R_p^i, R_e^j, $i = 1, \ldots, \mu$; $j = 1, \ldots, \nu$ are convex and compact and so is $\Phi(t_1, t_o)x_o + \sum_{j=1}^{\nu} R_e^j$. As Ω is closed and convex, then $\Omega - \sum_{i=1}^{\mu} R_p^i$ is closed convex. In view of definition 5.1, the system (4.1) is max-min Ω-controllable on T iff

$$\Phi(t_1, t_o)x_o + \sum_{j=1}^{\nu} R_e^j \subseteq \Omega - \sum_{i=1}^{\mu} R_p^i . \tag{5.2}$$

By lemma 3.1, (5.2) holds iff for all $g \in \mathbb{R}^n$, so we have

$$\langle g, \Phi(t_1, t_o)x_o\rangle + \sup_{\substack{v_j\in V_j^{\sigma_j} \\ j=1,\ldots,\nu}} \langle g, \sum_{j=1}^{\nu} \Lambda_e^j(v_j)\rangle$$
$$\leq \sup_{y\in\Omega} \langle g, y\rangle + \sup_{\substack{u_i\in U_i^{\rho_i} \\ i=1,\ldots,\mu}} \langle g, -\sum_{i=1}^{\mu} \Lambda_p^i(u_i)\rangle .$$

Similar to the proof of theorem 4.1, (5.1) then follows.

Corollary 5.1. If Ω is an ε ball $B(y_0, \varepsilon) = \{y : \|y - y_0\|_2 \leq \varepsilon\}$, then the system (4.1) is max-min Ω-controllable on T iff for all $g \in \mathbb{R}^n$, we have

$$\begin{aligned} \langle g, \Phi(t_1, t_0)x_0\rangle &\leq \langle g, y_0\rangle + \varepsilon\|g\|_2 \\ &+ \sum_{i=1}^{\mu} \rho_i \left(\int_{t_0}^{t_1} \|B_p^{i*}(\tau)\, \Phi^*(t_1, \tau)g\|_{s_i}^{q_i} d\tau\right)^{1/q_i} \\ &- \sum_{j=1}^{\nu} \sigma_j \left(\int_{t_0}^{t_1} \|B_e^{j*}(\tau)\, \Phi^*(t_1, \tau)g\|_{s_i'}^{q_i'} d\tau\right)^{1/q_i'} \end{aligned} \tag{5.3}$$

Proof: Since $B(y_0, \varepsilon) = y_0 + B(0, \varepsilon)$, then

$$\sup_{y \in B(y_0, \varepsilon)} \langle g, y\rangle = \langle g, y_0\rangle + \sup_{y \in B(0, \varepsilon)} \langle g, y\rangle = \langle g, y_0\rangle + \varepsilon\|g\|_2$$

and the result follows from theorem 5.1.

Remark 5.1. Corollary 5.1 gives a characterization for an "approximate" controllable system.

Remark 5.2. If Ω is convex and compact, then $\sup_{y \in \Omega} \langle g, y\rangle = \max_{y \in \Omega} \langle g, y\rangle$. Further, if Ω is unbounded such that $\sup_{y \in \Omega} \langle g, y\rangle = \infty$ holds for all $g \in \mathbb{R}^n$ with $\|g\|_2 = 1$, then the system (4.1) is always max-min Ω-controllable on T.

If the target set Ω is some subspace of $\mathbb{R}^n$, we want to derive a more explicit form than (5.1) to check controllability of the system. Let r be the dimension of Ω and let P_r be a projection to $\Omega^\perp$ along Ω. Then, P_r is linear and continuous and can be represented by a $n \times n$ matrix.

Thus, $P_r R_p^i$, $P_r R_e^j$, $i = 1, \ldots, \mu;\ j = 1, \ldots, \nu$, are convex and compact. Moreover, $x(t_1) \in \Omega$ iff $P_r x(t_1) = 0$.

Similar to theorem 5.1, we have the following.

Theorem 5.2. If Ω is a subspace of $\mathbb{R}^n$ with dimension r , then the system (4.1) is max-min Ω-controllable on T iff for all $g \in \mathbb{R}^n$, we have

$$|\langle g, P_r \Phi(t_1, t_o)x_o\rangle| \leq \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_p^{i*}(\tau)\, \Phi^*(t_1, \tau)P_r^* g\|_{s_i}^{q_i} d\tau\right)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \left(\int_{t_o}^{t_1} \|B_e^{j*}(\tau)\, \Phi^*(t_1, \tau)P_r^* g\|_{s_i'}^{q_i'} d\tau\right)^{1/q_i'} . \quad (5.4)$$

Remark 5.3. If $r = n$, i.e. $\Omega = \mathbb{R}^n$, then the problem has no meaning. If $r = 0$, $\Omega = \{0\}$, then $P_o = I$, the identity matrix and hence (5.4) becomes (4.9).

Example. Consider the game problem in $\mathbb{R}$ (see Pontryagin [10])

$$\begin{aligned} &\ddot{x} + \alpha\dot{x} = u \qquad x(0) = x_o \qquad \dot{x}(0) = 0 \\ &\ddot{y} + \beta\dot{y} = v \qquad y(0) = y_o \qquad \dot{y}(0) = 0 \end{aligned} \qquad (5.5)$$

on $T = [0, 1]$, such that

$$\left(\int_o^1 |u|^2 dt\right)^{\frac{1}{2}} \leq \rho \quad , \quad \left(\int_o^1 |v|^2 dt\right)^{\frac{1}{2}} \leq \sigma$$

where $\alpha, \beta, \rho, \sigma$ are positive, and $x_o \neq y_o$; x denotes the state of pursuer while y denotes that of evader.

The system (5.5) can be written in the form

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5.6}$$

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \quad . \tag{5.7}$$

Then, the state transition matrix for (5.6) is

$$\Phi_p(t, \tau) = \begin{bmatrix} 1 & \frac{1}{\alpha}(1 - e^{-\alpha(t-\tau)}) \\ 0 & e^{-\alpha(t-\tau)} \end{bmatrix} \tag{5.8}$$

while that for (5.7) is

$$\Phi_e(t, \tau) = \begin{bmatrix} 1 & \frac{1}{\beta}(1 - e^{-\beta(t-\tau)}) \\ 0 & e^{-\beta(t-\tau)} \end{bmatrix} \quad . \tag{5.9}$$

Consider the transformation

$$z = \Phi_p(1, t)\begin{bmatrix} x \\ \dot{x} \end{bmatrix} - \Phi_e(1, t)\begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{5.10}$$

then

$$\dot{z} = \Phi_p(1, t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}u - \Phi_e(1, t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}v$$

i.e.
$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha}(1 - e^{-\alpha(1-t)}) \\ e^{-\alpha(1-t)} \end{bmatrix}u - \begin{bmatrix} \frac{1}{\beta}(1 - e^{-\beta(1-t)}) \\ e^{-\beta(1-t)} \end{bmatrix}v \tag{5.11}$$

also

$$z(0) = \begin{bmatrix} x_o - y_o \\ 0 \end{bmatrix} \quad \text{and} \quad x(1) = y(1) \quad \text{iff} \quad z_1(1) = 0 \ .$$

Here, the state transition matrix of (5.11) is $\Phi(t, \tau) = I$. As we only want $x(1) = y(1)$, i.e. $z_1(1) = 0$, so that $\Omega = \{ \begin{pmatrix} 0 \\ w \end{pmatrix} : w \in \mathbb{R} \} \subseteq \mathbb{R}^2$ and $P_1 = [1 \quad 0]$.

To test for controllability, consider $g \in \mathbb{R}$, $\langle g, P_1 z_o \rangle = g(x_o - y_o)$

and

$$\int_0^1 \|B_p^*(\tau)\,\Phi^*(1,\tau)P_1^*\,g\|_2^2\,d\tau$$

$$= \int_0^1 \frac{1}{\alpha^2}\,(1 - e^{-\alpha(1-\tau)})^2 g^2\,d\tau$$

$$= \frac{g^2}{\alpha^2}\,(1 - \frac{3}{2\alpha} + \frac{2}{\alpha}e^{-\alpha} - \frac{1}{2\alpha}e^{-2\alpha}) \geq 0$$

and

$$\int_0^1 \|B_e^*(\tau)\,\Phi^*(1,\tau)P_1^*g\|_2^2\,d\tau = \frac{g^2}{\beta^2}\,(1 - \frac{3}{2\beta} + \frac{2}{\beta}e^{-\beta} - \frac{1}{2\beta}e^{-2\beta})$$

$$= \frac{g^2}{\beta^3}\,(\beta - \frac{3}{2} + 2e^{-\beta} - \frac{1}{2}e^{-2\beta})\;.$$

Let

$$f(w) = w - \frac{3}{2} + 2e^{-w} - \frac{1}{2}e^{-2w} \geq 0\;,$$

then

$$f'(w) = 1 - 2e^{-w} + e^{-2w} = (1 - e^{-w})^2 \geq 0\;.$$

Thus, $f(w)$ is increasing for $w \neq 0$ and so is its square root. Now condition (5.4) becomes

$$|g|\;|x_o - y_o| \leq \frac{\rho}{\alpha^{3/2}}\,|g|\,(\alpha - \frac{3}{2} + 2e^{-\alpha} - \frac{1}{2}e^{-2\alpha})^{\frac{1}{2}}$$
$$- \frac{\sigma}{\beta^{3/2}}\,|g|\,(\beta - \frac{3}{2} + 2e^{-\alpha} - \frac{1}{2}e^{-2\beta})^{\frac{1}{2}}\;. \qquad (5.12)$$

Two cases of interest arise:

(i) If $\beta \geq \alpha$ and $\frac{\sigma}{\beta^{3/2}} \geq \frac{\rho}{\alpha^{3/2}}$ then (5.12) cannot hold and hence the system (5.5) is not max-min null-controllable on $[0, 1]$.

(ii) Since $f(\alpha) > f(\beta)$ if $\alpha > \beta$ so that if $\alpha > \beta$ such that

$$f^{\frac{1}{2}}(\alpha) - f^{\frac{1}{2}}(\beta) \geq \frac{\alpha^{3/2}}{\rho}|x_o - y_o| \quad \text{and} \quad \frac{\rho}{\alpha^{3/2}} \geq \frac{\sigma}{\beta^{3/2}}$$

then (5.12) holds for all $g \in \mathbb{R}$. Therefore, the system (5.5) is max-min null-controllable on $[0, 1]$.

VI. Local Controllability, Observability and Duality

In this section, we only consider the cooperative problem, described by a linear differential equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + \sum_{i=1}^{\mu} B_i(t)u_i(t) \\ y_j(t) &= C_j(t)x(t) , \quad j = 1, \ldots, \mu . \end{aligned} \tag{6.1}$$

on the fixed time interval $T = [t_o, t_1]$, where $x, A(t), B_i(t), u_i(t)$ are as before, y_j denotes an m_j dimension output vector while $C_j(t)$ is an $m_j \times n$ matrix with their components being Lebesque integrable. Now, we restrict our study to $L_{2,2}(T)$ i.e. $p_i = q_i = r_i = s_i = 2$. The input constraints

$$U_i^{\rho_i} = \{u_i : (\int_{t_o}^{t_1} \|u_i(t)\|_2^2 \, dt)^{\frac{1}{2}} \leq \rho_i \} ; \quad i = 1, \ldots, \mu . \tag{6.2}$$

We define the formal adjoint system as

$$\begin{aligned} \dot{z}(t) &= -A^*(t)z(t) + \sum_{i=1}^{\mu} C_i^*(t)v_i(t) \\ w_j(t) &= B_j^*(t)z(t) , \quad j = 1, \ldots, \mu , \end{aligned} \tag{6.3}$$

on T , where the restriction of v_i is to be determined later.

Definition 6.1. The system (6.1) is said to be locally null-controllable on T if there is a neighborhood N_o of the origin such that it is null-controllable on T at each point of N_o .

Remark 6.1. In view of definition 6.1, the neighborhood N_o can be taken to be an ε-neighborhood $N_o(\varepsilon)$.

Theorem 6.1. The system (6.1) is locally null-controllable on T iff there exists an $\varepsilon > 0$ and for all $g \in \mathbb{R}^n$, we have

$$\varepsilon\|\Phi^*(t_1, t_o)g\|_2 \leq \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_i^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau\right)^{\frac{1}{2}} . \tag{6.4}$$

Proof: The system (6.1) is locally null-controllable on T iff $\exists$ ε-neighborhood $N_o(\varepsilon)$ such that for each $x_o \in N_o(\varepsilon)$ and for all $g \in \mathbb{R}^n$, we have

$$|\langle g, \Phi(t_1, t_o)x_o\rangle| \leq \sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_i^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau\right)^{\frac{1}{2}} . \tag{6.5}$$

But this is equivalent to the supremum of the left hand side of (6.5)

$$\begin{aligned} \sup_{x_o \in N_o(\varepsilon)} |\langle g, \Phi(t_1, t_o)x_o\rangle| &= \sup_{x_o \in N_o(\varepsilon)} |\langle \Phi^*(t_1, t_o)g, x_o\rangle| \\ &= \varepsilon\|\Phi^*(t_1, t_o)g\|_2 \end{aligned}$$

is less than or equal to the right hand side of (6.5). Thus, (6.4) is established.

Corollary 6.1. The system (6.1) is locally null-controllable on T (i) iff for all $0 \neq g \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_i^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau\right)^{\frac{1}{2}} > 0 \tag{6.6}$$

and (ii) iff for all $0 \neq g \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{\mu} \int_{t_o}^{t_1} \|B_i^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau > 0 . \tag{6.7}$$

Proof: By theorem 6.1, the system (6.1) is locally null-controllable on T iff for all $g \in \mathbb{R}^n$, (6.4) holds. If $g \neq 0$, then since $\Phi(t_1, t_o)$ is non-singular,

$$\|\Phi^*(t_1, t_o)g\|_2 > 0 .$$

Thus (6.6) holds. Conversely, suppose (6.4) is false for any $\varepsilon > 0$, then for each n, there exists $g_n \neq 0$ such that

$$\sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_i^*(\tau)\Phi^*(t_1, \tau)g_n\|_2^2 \, d\tau \right)^{\frac{1}{2}} < \frac{1}{n} \|\Phi^*(t_1, t_o)g_n\|_2 . \tag{6.8}$$

Without loss of generality, we assume $\|g_n\|_2 = 1$ and hence $\exists$ subsequence $g_{n_k} \to \hat{g}$ with $\|\hat{g}\| = 1$. Passing to the limit as $k \to \infty$, (6.8) becomes

$$\sum_{i=1}^{\mu} \rho_i \left(\int_{t_o}^{t_1} \|B_i^*(\tau)\Phi^*(t_1, \tau)\hat{g}\|_2^2 \, d\tau \right)^{\frac{1}{2}} \leq 0 .$$

Thus, (6.6) cannot hold for all $g \neq 0$. Since $\rho_i > 0$, $i = 1, \ldots, \mu$, then (6.6) holds iff there exists some index j such that

$$\rho_j \left(\int_{t_o}^{t_1} \|B_j^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau \right)^{\frac{1}{2}} > 0$$

or

$$\int_{t_o}^{t_1} \|B_j^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau > 0$$

and hence (6.7) holds.

Define $$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_\mu(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_\mu(t) \end{bmatrix}, \quad C(t) = \begin{bmatrix} C_1(t) \\ \vdots \\ C_\mu(t) \end{bmatrix} . \tag{6.9}$$

$$B(t) = [B_1(t), \ldots, B_\mu(t)] .$$

Corollary 6.2. The system (6.1) is locally null-controllable on

T iff for all $0 \neq g \in \mathbb{R}^n$, we have

$$\int_{t_o}^{t_1} \|B^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau > 0 \tag{6.10}$$

and thus, the matrix

$$W(t_o, t_1) = \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)B^*(\tau)\Phi^*(t_1, \tau)d\tau \tag{6.11}$$

is positive definite.

Proof: For $g \neq 0$

$$\begin{aligned} g^*W(t_o, t_1)g &= \int_{t_o}^{t_1} \|B^*(\tau)\Phi^*(t_o, \tau)g\|_2^2 \, d\tau \\ &= \int_{t_o}^{t_1} \left\| \begin{matrix} B_1^*(\tau)\Phi^*(t_1, \tau)g \\ \vdots \\ B_\mu^*(\tau)\Phi^*(t_1, \tau)g \end{matrix} \right\|_2^2 d\tau \\ &= \sum_{i=1}^{\mu} \int_{t_o}^{t_1} \|B_i^*(\tau)\Phi^*(t_1, \tau)g\|_2^2 \, d\tau \, . \end{aligned}$$

Hence, $W(t_o, t_1)$ is positive definite iff (6.10) holds and iff (6.7) holds.

Remark 6.2. It is well-known that the system (6.1) without input constraints is completely controllable on T iff the matrix $W(t_o, t_1)$ is positive definite. Consequently, the concept of local null-controllability with input constraints coincides with that of complete controllability in the sense when ρ_i, $i = 1, \ldots, \mu$ all tend to infinity.

Now, we consider the observability of the system.

Definition 6.2. The system (6.1) is said to be observable on T if the observations $y_j(t)$, $t \in T$, $j = 1, \ldots, \mu$, together with $u_i \in U_i^{\rho_i}$, $i = 1, \ldots, \mu$ can determine the initial state $x(t_0)$.

Theorem 6.2. The system (6.1) is observable on T iff the matrix

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi^*(\tau, t_1)C^*(\tau)C(\tau)\Phi(\tau, t_1)d\tau \tag{6.12}$$

is positive definite.

Proof: Omitted.

Remark 6.3. As inputs are known, input constraints will not directly affect the observability of the system.

The following theorem is an extension of the fundamental duality result due to Kalman for systems without input constraints.

Theorem 6.3. The system (6.1) is locally null-controllable on T iff the adjoint system (6.3) is observable on T. The system (6.1) is observable on T iff the adjoint system (6.3) is locally null-controllable on T.

Proof: Since $\Phi(t, \tau)$ is the state transition matrix for the system (6.1), then $\Phi^*(\tau, t)$ is the state transition matrix for the adjoint system (6.3) so that $W(t_0, t_1)$ is the observability matrix for the adjoint system (6.3) while $M(t_0, t_1)$ is the controllability matrix for the adjoint system (6.3). In view of Corollary 6.2 and theorem 6.2, the results follow.

Now, we want to determine the admissible control sets of the adjoint system. Define

$$V_\varepsilon = \left\{ \begin{array}{l} \sigma = (\sigma_1, \ldots, \sigma_\mu) : \sigma_i \geq 0, \ i = 1, \ldots, \mu \quad \text{and for all} \quad g \in \mathbb{R}^n, \\ \varepsilon \|\Phi(t_o, t_1)g\|_2 \leq \sum_{i=1}^{\mu} \sigma_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g\|_2^2 \, d\tau \right)^{\frac{1}{2}} . \end{array} \right\}, \tag{6.13}$$

the norm for σ

$$\|\sigma\|_2 = \left(\sum_{i=1}^{\mu} \sigma_i^2 \right)^{\frac{1}{2}} , \tag{6.14}$$

and

$$E_\sigma = \{\varepsilon > 0 : \varepsilon\|\Phi(t_o, t_1)g\|_2 \leq \sum_{i=1}^{\mu} \sigma_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g\|_2^2 d\tau \right)^{\frac{1}{2}}, \ \forall g \in \mathbb{R}^n . \} \tag{6.15}$$

In view of theorem 6.1, the adjoint system (6.3) is locally null-controllable on T iff for each $\varepsilon > 0$, $V_\varepsilon \neq \phi$; or iff for some $\sigma = (\sigma_1, \ldots, \sigma_\mu)$ such that $\sigma_i \geq 0$, $E_\sigma \neq \phi$.

Lemma 6.1. If $V_\varepsilon \neq \phi$, $\varepsilon > 0$, then there exists an element $\hat{\sigma}_\varepsilon = (\hat{\sigma}_1, \ldots, \hat{\sigma}_\mu) \in V_\varepsilon$ such that

$$\|\hat{\sigma}_\varepsilon\|_2 = \inf\{ \|\sigma\|_2 : \sigma \in V_\varepsilon \} \tag{6.16}$$

and thus $\exists \ o \neq \hat{g} \in \mathbb{R}^n$ with $\|\hat{g}\|_2 = 1$ such that

$$\varepsilon\|\Phi(t_o, t_1)\hat{g}\|_2 = \sum_{i=1}^{\mu} \hat{\sigma}_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)\hat{g}\|_2^2 \, d\tau \right)^{\frac{1}{2}} . \tag{6.17}$$

Proof: Let $\lambda = \inf\{\|\sigma\|_2 : \sigma \in V_\varepsilon\}$. Then $\exists\{\sigma^n\} \subset V_\varepsilon$ such that $\|\sigma^n\|_2 \to \lambda$. Thus, $\{\sigma^n\}$ is a bounded sequence in $\mathbb{R}^\mu$. There exists a convergent subsequence $\sigma^{n_k} \to \hat{\sigma}_\varepsilon = (\hat{\sigma}_1, \ldots, \hat{\sigma}_\mu)$ and so $\sigma_i^{n_k} \to \hat{\sigma}_i$, $i = 1, \ldots, \mu$. By uniqueness of the limit, we have $\|\hat{\sigma}_\varepsilon\|_2 = \lambda$. Since $\sigma^{n_k} \in V_\varepsilon$, we have $\sigma_i^{n_k} \geq 0$ and for all $g \in \mathbb{R}^n$

$$\varepsilon\|\Phi(t_o, t_1)g\|_2 \leq \sum_{i=1}^{\mu} \sigma_i^{n_k} \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g\|_2^2 d\tau\right)^{\frac{1}{2}} .$$

Passing to limit as $k \to \infty$, we have $\hat{\sigma}_i \geq 0$ and

$$\varepsilon\|\Phi(t_o, t_1)g\|_2 \leq \sum_{i=1}^{\mu} \hat{\sigma}_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g\|_2^2 d\tau\right)^{\frac{1}{2}} \tag{6.18}$$

for all $g \in \mathbb{R}^n$, so that $\hat{\sigma}_\varepsilon \in V_\varepsilon$. On the other hand, we choose $\theta^n = (\theta_1^n, \ldots, \theta_\mu^n)$ such that $\theta_i^n < \sigma_i$ and $\theta_i^n \to \sigma_i$ as $n \to \infty$, $i = 1, \ldots, \mu$. Thus $\|\theta^n\|_2 \to \|\hat{\sigma}_\varepsilon\|_2$ with $\|\theta^n\|_2 < \|\hat{\sigma}_\varepsilon\|_2$, so that $\theta^n \notin V_\varepsilon$ and for each n $\exists\, o \neq g_n \in \mathbb{R}^n$ such that

$$\varepsilon\|\Phi(t_o, t_1)g_n\|_2 > \sum_{i=1}^{\mu} \theta_i^n \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g_n\|_2^2 d\tau\right)^{\frac{1}{2}} .$$

Without loss of generality, we can assume $\|g_n\|_2 = 1$ and $\exists$ subsequence $g_{n_k} \to \hat{g}$ with $\|\hat{g}\|_2 = 1$. Passing to the limit as $k \to \infty$, we have

$$\varepsilon\|\Phi(t_o, t_1)\hat{g}\|_2 \geq \sum_{i=1}^{\mu} \hat{\sigma}_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)\hat{g}\|_2^2 d\tau\right)^{\frac{1}{2}} . \tag{6.19}$$

From (6.18) with $g = \hat{g}$, we see that (6.17) is satisfied.

Lemma 6.2. If $E_\sigma \neq \phi$, $\sigma_i \geq 0$, then there exists $\hat{\varepsilon}_\sigma \in E_\sigma$ such that

$$\hat{\varepsilon}_\sigma = \sup\{\varepsilon, \varepsilon \in E_\sigma\} \tag{6.20}$$

and $\exists\, o \neq \bar{g} \in \mathbb{R}^n$ with $\|\bar{g}\|_2 = 1$ such that

$$\hat{\varepsilon}_\sigma\|\Phi(t_o, t_1)\bar{g}\|_2 = \sum_{i=1}^{\mu} \sigma_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)\bar{g}\|_2^2 d\tau\right)^{\frac{1}{2}} . \tag{6.21}$$

Proof: Choose $\varepsilon_n \in E_\sigma$ such that $\varepsilon_n \to \hat{\varepsilon}_\sigma$, so that $\hat{\varepsilon}_\sigma > 0$

and for each n and all $g \in \mathbb{R}^n$, we have

$$\varepsilon_n \|\Phi(t_o, t_1)g\|_2 \leq \sum_{i=1}^{\mu} \sigma_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g\|_2^2 \, d\tau \right)^{\frac{1}{2}} .$$

Passing to the limit, we have $\hat{\varepsilon}_\sigma \in E_\sigma$. On the other hand, choose $\delta_n > \hat{\varepsilon}_\sigma$ with $\delta_n \to \hat{\varepsilon}_\sigma$, so that $\delta_n \notin E_\sigma$ and for each n, $\exists\, 0 \neq g_n \in \mathbb{R}^n$ such that

$$\delta_n \|\Phi(t_o, t_1)g_n\|_2 > \sum_{i=1}^{\mu} \sigma_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g_n\|_2^2 \, d\tau \right)^{\frac{1}{2}} .$$

Without loss of generality, we assume $\|g_n\|_2 = 1$ and so there exists subsequence $g_{n_k} \to \bar{g}$ with $\|\bar{g}\|_2 = 1$. Passing to the limit, we have

$$\varepsilon_\sigma \|\Phi(t_o, t_1)\bar{g}\|_2 \geq \sum_{i=1}^{\mu} \sigma_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)\bar{g}\|_2^2 \, d\tau \right)^{\frac{1}{2}} .$$

Since $\hat{\varepsilon}_\sigma \in E_\sigma$, therefore (6.21) holds.

Theorem 6.4. If $V_\varepsilon \neq \phi$, $\varepsilon > 0$ and

$$\bar{\varepsilon} = \sup\{\delta > 0 : \delta \in E_{\hat{\sigma}_\varepsilon}\} , \tag{6.22}$$

then $\bar{\varepsilon} \in E_{\hat{\sigma}_\varepsilon}$ and $\bar{\varepsilon} = \varepsilon$.

Proof: By lemma 6.1, $\hat{\sigma}_\varepsilon \in V_\varepsilon$ and thus $\varepsilon \in E_{\hat{\sigma}_\varepsilon}$. By lemma 6.2, $\bar{\varepsilon} \in E_{\hat{\sigma}_\varepsilon}$ and $\bar{\varepsilon} \geq \varepsilon$, so that for all $g \in \mathbb{R}^n$, we have

$$\varepsilon \|\Phi(t_o, t_1)g\|_2 \leq \bar{\varepsilon} \|\Phi(t_o, t_1)g\|_2 \leq \sum_{i=1}^{\mu} \hat{\sigma}_i \left(\int_{t_o}^{t_1} \|C_i(\tau)\Phi(\tau, t_1)g\|_2^2 \, d\tau \right)^{\frac{1}{2}} .$$

In view of (6.17) with $g = \hat{g}$, we see that $\varepsilon = \bar{\varepsilon}$.

Remark 6.4. If $E_\sigma \neq \phi$, $\sigma_i \geq 0$, then $\exists\, \grave{\varepsilon}_\sigma \in E_\sigma$ and thus

$\sigma \in V_{\hat{\varepsilon}_\sigma}$ and $\bar{\sigma} \in V_{\hat{\varepsilon}_\sigma}$ is of minimal norm. As theorem 6.4, one would expect $\|\bar{\sigma}\|_2 = \|\sigma\|_2$. However, this case holds only if $\mu = 1$. The following is a counterexample.

Example. The adjoint system (6.3) on $T = [0, 1]$ with $\mu = 2$. Let $A = I = C_1$, $C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\sigma_1 > 0, \sigma_2 > 0$. The state transition matrix $\Phi^*(\tau, t) = e^{\tau - t} I$.

Note that for $\varepsilon > 0$ and $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathbb{R}^2$, $\varepsilon\|\Phi(0, 1)g\|_2 = \varepsilon e^{-1}\|g\|_2$,

$$\int_0^1 \|C_1(\tau)\Phi(\tau, 1)g\|_2^2 \, d\tau = \int_0^1 e^{2(\tau-1)} \|g\|_2^2 \, d\tau = \tfrac{1}{2}(1 - e^{-2})\|g\|_2^2 ,$$

$$\int_0^1 \|C_2(\tau)\Phi(\tau, 1)g\|_2^2 \, d\tau = \int_0^1 e^{2(\tau-1)} |g_1|^2 \, d\tau = \tfrac{1}{2}(1 - e^{-2})|g_1|^2 .$$

Let $\varepsilon = \sigma_1$, then $\varepsilon < \frac{e}{\sqrt{2}} (1 - e^{-2})^{\frac{1}{2}} \sigma_1 = \frac{1}{\sqrt{2}} (e^2 - 1)^{\frac{1}{2}} \sigma_1$. Thus $E_\sigma \neq \phi$ and the system is locally null-controllable iff the original system is observable on T . Obviously, $\hat{\varepsilon}_\sigma = \frac{1}{\sqrt{2}} (e^2 - 1)^{\frac{1}{2}} \sigma_1$ and $\bar{\sigma} = (\sigma_1, 0)$. Hence $\|\bar{\sigma}\|_2 < \|\sigma\|_2$.

VII. Discrete time system

In this section, we consider linear systems described by difference equations. Since the results and proofs are similar to the continuous time case, we just state them without giving detail proofs for brevity.

Consider the general game system described by the difference equation

$$x(k+1) = A(k)x(k) + \sum_{i=1}^{\mu} B_p^i(k)u_i(k) + \sum_{j=1}^{\nu} B_e^j(k)v_j(k), \quad k = 0, 1, \ldots, \gamma, \qquad (7.1)$$

$$x(0) = x_o ,$$

where $A(k)$, $B_p^i(k)$, $B_e^j(k)$, $x(k)$ are of appropriate dimensions. The solution of (7.1) can be expressed as

$$x(k) = \Phi(k, 0)x_o + \sum_{i=1}^{\mu} \sum_{\ell=1}^{k=1} \Phi(k, \ell + 1)B_p^i(\ell)u_i(\ell)$$

$$+ \sum_{j=1}^{\nu} \sum_{\ell=0}^{k-1} \Phi(k, \ell + 1)B_e^j(\ell)v_j(\ell)$$

$$k \geq 1 , \qquad (7.2)$$

where the matrix defined by

$$\Phi(i, j) = \begin{cases} A(i-1)A(i-2) \ldots A(0), & \text{if } i > j \\ I & , \text{if } i = j \end{cases} \qquad (7.3)$$

is called the transition matrix of (7.1). $\Phi(\gamma, 0)$ is non-singular iff A_k, $k = 0, \ldots, \gamma-1$ are non-singular.

For convenience, here we use similar notations as in the continuous time case.

Define

$$\|u_i(k)\|_{r_i} = \left(\sum_{j=1}^{h_i} |u_i^j(k)|^{r_i}\right)^{1/r_i} \qquad 1 \leq r_i < \infty ,$$

$$\|u_i(k)\|_{\infty} = \max_{j=1,\ldots,h_i} \{|u_i^j(k)|\} ,$$

$$\big|\|u_i(k)\|_{r_i}\big|_{p_i} = \left(\sum_{k=0}^{\gamma-1} \|u_i(k)\|_{r_i}^{p_i}\right)^{1/p_i} \qquad 1 \leq p_i < \infty ,$$

$$\left| \| u_i(k) \|_{r_i} \right|_\infty = \max_{k=0,\dots,\gamma-1} \{ \| u_i(k) \|_{r_i} \} . \tag{7.4}$$

Thus, the sets

$$U_i^{\rho_i} = \{ \{u_i(k)\} : \left| \| u_i(k) \|_{r_i} \right|_{p_i} \leq \rho_i \} , \quad i = 1, \dots, \mu \tag{7.5}$$

$$V_j^{\sigma_j} = \{ \{v_j(k)\} : \left| \| v_j(k) \|_{r'_j} \right|_{p'_j} \leq \sigma_j \} , \quad j = 1, \dots, \nu \tag{7.6}$$

are convex and compact in the respective finite dimensional spaces. Let

$$\Lambda_p^i(u_i(k)) = \sum_{k=0}^{\gamma-1} \Phi(\gamma, k+1) B_p^i(k) u_i(k) \qquad i = 1, \dots, \mu , \tag{7.7}$$

$$\Lambda_e^j(v_j(k)) = \sum_{k=0}^{\gamma-1} \Phi(\gamma, k+1) B_e^j(k) v_j(k) \qquad j = 1, \dots, \nu , \tag{7.8}$$

$$R_p^i = \{ \Lambda_p^i(u_i(k)) : \{u_i(k)\} \in U_i^{\rho_i} \} \qquad i = 1, \dots, \mu , \tag{7.9}$$

$$R_e^j = \{ \Lambda_e^j(v_j(k)) : \{v_j(k)\} \in V_j^{\sigma_j} \} \qquad j = 1, \dots, \nu , \tag{7.10}$$

then Λ_p^i, Λ_e^j are continuous linear compact operators and hence R_p^i, R_e^j, $i = 1, \dots, \mu$; $j = 1, \dots, \nu$ are compact.

For brevity, denote

$$\{0, k\} \equiv \{0, 1, \dots, k\} . \tag{7.11}$$

Definition 7.1. The discrete time system (7.1) is said to be max-min Ω-controllable on $\{0, \gamma\}$ if for each announced evaders' controls $\{v_j(k)\} \in V_j^{\sigma_j}$, $j = 1, \dots, \nu$, there exist pursuers' controls $\{u_i(k)\} \in U_i^{\rho_i}$, $i = 1, \dots, \mu$, which steer the initial state x_0 to Ω in γ-th step, i.e. $x(\gamma) \in \Omega$.

Theorem 7.1. If Ω is closed convex, then the discrete time system is max-min Ω-controllable on $\{0, \gamma\}$ iff for all $g \in \mathbb{R}^n$, we have

$$\langle g, \Phi(\gamma, 0)x_o\rangle \leq \sup_{y\in\Omega} \langle g, y\rangle + \sum_{i=1}^{\mu} \rho_i \Big(\sum_{k=o}^{\gamma-1} \|B_p^{i*}(k)\Phi^*(\gamma, k+1)g\|_{s_i}^{q_i}\Big)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \Big(\sum_{k=o}^{\gamma-1} \|B_e^{j*}(k)\Phi^*(\gamma, k+1)g\|_{s_i'}^{q_i'}\Big)^{1/q_i'} . \quad (7.12)$$

If $\Omega = B(y_o, \varepsilon)$ in $\mathbb{R}^n$, then

$$\sup_{y\in\Omega} \langle g, y\rangle = \langle g, y_o\rangle + \varepsilon\|g\|_2 . \quad (7.13)$$

Furthermore, if Ω is a subspace of $\mathbb{R}^n$ with dimension r and P_r is a projection to $\Omega^\perp$ along Ω, then the necessary and sufficient condition for the discrete time system to be max-min Ω-controllable on $\{0, \gamma\}$ becomes

$$|\langle g, P_r\Phi(\gamma, 0)x_o\rangle| \leq \sum_{i=1}^{\mu} \rho_i \Big(\sum_{k=o}^{\gamma-1} \|B_p^{i*}(k)\Phi^*(\gamma, k+1)P_r^*g\|_{s_i}^{q_i}\Big)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \Big(\sum_{k=o}^{\gamma-1} \|B_e^{j*}(k)\Phi^*(\gamma, k+1)P_r^*g\|_{s_i'}^{q_i'}\Big)^{1/q_i'} \quad (7.14)$$

for all $g \in \mathbb{R}^n$.

Remark 7.1. If $\nu = 0$, then the system is a cooperative control problem, not a game.

Next, we consider the discrete time system with outputs.

$$\begin{aligned} x(k+1) &= A(k)x(k) + \sum_{i=1}^{\mu} B_i(k)u_i(k) \qquad k = 0, \ldots, \gamma-1, \\ y_j(k) &= C_j(k)x(k) \qquad\qquad j = 1, \ldots, \mu , \end{aligned} \quad (7.15)$$

where $\{u_i(k)\} \in U_i^{\rho_i}$, $\rho_i > 0$, $i = 1, \ldots, \mu$, with $p_i = q_i = r_i = s_i = 2$.

Define the formal adjoint system as follows

$$z(k-1) = A^*(k)z(k) + \sum_{i=1}^{\mu} C_i^*(k)v_i(k) \qquad k = 1, \ldots, \gamma ,$$
$$w_j(k) = B_j^*(k)z(k) \qquad j = 1, \ldots, \mu , \tag{7.16}$$

where the input constraints will be determined later.

Definition 7.2. The discrete time system (7.15) is said to be locally null-controllable on $\{0, \gamma\}$ if there exists a neighborhood N_o of origin such that for each $x_o \in N_o$, it is null-controllable on $\{0, \gamma\}$.

Definition 7.3. The discrete time system (7.15) is said to be locally null-reachable on $\{0, \gamma\}$ if there exists a neighborhood N_o of origin such that for each $x_n \in N_o$, $\exists \{u_i(k)\} \in U_i^{\rho_i}$ for which x_n can be reached from the origin.

Let

$$y(k) = \begin{bmatrix} y_1(k) \\ \vdots \\ y_\mu(k) \end{bmatrix}, \quad u(k) = \begin{bmatrix} u_1(k) \\ \vdots \\ u_\mu(k) \end{bmatrix}, \quad C(k) = \begin{bmatrix} C_1(k) \\ \vdots \\ C_\mu(k) \end{bmatrix}, \quad v(k) = \begin{bmatrix} v_1(k) \\ \vdots \\ v_\mu(k) \end{bmatrix}$$
$$B(k) = [B_1(k), \ldots, B_\mu(k)] . \tag{7.17}$$

Theorem 7.2. The discrete time system (7.15) is locally null-controllable on $\{0, \gamma\}$ iff $\exists\, \varepsilon > 0$ and for all $g \in \mathbb{R}^n$, we have

$$\varepsilon\|\Phi^*(\gamma, 0)g\|_2 \leq \sum_{i=1}^{\mu} \rho_i \Big(\sum_{k=0}^{\gamma-1} \|B_i^*(k)\Phi^*(\gamma, k+1)g\|_2^2\Big)^{\frac{1}{2}} . \tag{7.18}$$

Theorem 7.3. The discrete time system (7.15) is locally null-reachable on $\{0, \gamma\}$ iff (i) $\exists\, \varepsilon > 0$ and for all $g \in \mathbb{R}^n$, we have

$$\varepsilon\|g\|_2 \leq \sum_{i=1}^{\mu} \rho_i \Big(\sum_{k=0}^{\gamma-1} \|B_i^*(k)\Phi^*(\gamma, k+1)g\|_2^2\Big)^{\frac{1}{2}} , \tag{7.19}$$

or (ii) for all $g \neq 0$, we have

$$\sum_{i=1}^{\mu} \rho_i \Big(\sum_{k=0}^{\gamma-1} \|B_i^*(k)\Phi^*(\gamma, k+1)g\|_2^2\Big)^{\frac{1}{2}} > 0 , \tag{7.20}$$

or (iii) for all $g \neq 0$, we have

$$\sum_{i=1}^{\mu} \sum_{k=0}^{\gamma-1} \|B_i^*(k)\Phi^*(\gamma, k+1)g\|_2^2 > 0 , \tag{7.21}$$

or (iv) for all $g \neq 0$, we have

$$\sum_{k=0}^{\gamma-1} \|B^*(k)\Phi^*(\gamma, k+1)g\|_2^2 > 0 , \tag{7.22}$$

or (v) the controllability matrix

$$W(0, \gamma) = \sum_{k=0}^{\gamma-1} \Phi(\gamma, k+1)B(k)B^*(k)\Phi^*(\gamma, k+1) \tag{7.23}$$

is positive definite.

Remark 7.2. From (7.18), (7.19), we see that locally null-controllability of system (7.15) can be implied by locally null-reachability, but not the converse. However, if $\Phi(\gamma, 0)$ is non-singular, then the two concepts are equivalent.

Definition 7.4. The discrete time system (7.15) is said to be observable on $\{0, \gamma\}$ if given the observations $y(k)$ and $u(k)$, $k = 0,1,\ldots,\gamma$, the initial state x_o can be determined.

Definition 7.5. The discrete time system (7.15) is said to be reconstructible on $\{0, \gamma\}$ if given the observations $y(k)$ and $u(k)$, $k = 0, 1, \ldots, \gamma$, the final state $x(\gamma)$ can be determined.

Remark 7.3. In the system (7.15)

$$x(\gamma) = \Phi(\gamma, 0)x_0 + \sum_{k=0}^{\gamma-1} \Phi(\gamma, k+1)B(k)u(k) \tag{7.24}$$

we see that observability implies reconstructibility but not the converse if $\Phi(\gamma, 0)$ is singular.

Theorem 7.4. The discrete time system (7.15) is observable on $\{0, \gamma\}$ iff the observability matrix

$$M(0, \gamma) = \sum_{k=0}^{\gamma} \Phi^*(k, 0)C^*(k)C(k)\Phi(k, 0) \tag{7.25}$$

is positive definite.

Refering back to the discrete time adjoint system (7.16), its solution is expressed as

$$z(0) = \Phi^*(\gamma+1, 1)z(\gamma) + \sum_{k=1}^{\gamma} \Phi^*(k, 1)C^*(k)v(k) . \tag{7.26}$$

We see that reconstructibility implies observability, but not the converse if $\Phi(\gamma + 1, 1)$ is singular. Furthermore, the system (7.16) is locally null-controllable on $\{0, \gamma\}$ iff $\exists\, \varepsilon > 0$ and for all $g \in \mathbb{R}^n$, we have

$$\varepsilon \|g\|_2 \leq \sum_{i=1}^{\mu} \sigma_i \Big(\sum_{k=1}^{\gamma} \|c_i(k)\Phi(k, 1)g\|_2^2\Big)^{\frac{1}{2}} \;; \tag{7.27}$$

while the system (7.16) is locally null-reachable on $\{0, \gamma\}$ iff $\exists\, \varepsilon > 0$ and for all $g \in \mathbb{R}^n$, we have

$$\varepsilon\|\Phi(\gamma + 1, 1)g\|_2 \leqslant \sum_{i=1}^{\mu} \sigma_i \left(\sum_{k=1}^{\gamma} \|C_i(k)\Phi(k, 1)g\|_2^2\right)^{\frac{1}{2}} . \tag{7.28}$$

In view of (7.27), (7.28), we see that, locally null-controllability implies locally null-reachability. These results are almost the opposite of one another. Thus, we have the important duality result, not quite the same as the continuous time case:

Theorem 7.5. The discrete time system (7.15) is locally null reachable on $\{0, \gamma\}$ iff its adjoint system (7.16) is reconstructible on $\{0, \gamma - 1\}$ while the discrete time system (7.15) is observable on $\{1, \gamma\}$ iff its adjoint system (7.16) is locally null controllable on $\{0, \gamma\}$.

Next, we want to determine the input constraints in the adjoint system (7.16). For $\varepsilon > 0$, consider the sets

$$V_\varepsilon = \left\{ \begin{array}{l} (\sigma_1, \ldots, \sigma_\mu) : \sigma_i \geqslant 0 \text{ and for all } g \in \mathbb{R}^n, \text{ we have} \\ \varepsilon\|g\|_2 \leqslant \displaystyle\sum_{i=1}^{\mu} \sigma_i \left(\sum_{k=1}^{\gamma} \|C_i(k)\,\Phi(k, 1)g\|_2^2\right)^{\frac{1}{2}} \end{array} \right\}, \tag{7.29}$$

$$E_\sigma = \left\{ \begin{array}{l} \varepsilon > 0 \ : \ \varepsilon\|g\|_2 \leqslant \displaystyle\sum_{i=1}^{\mu} \sigma_i \left(\sum_{k=1}^{\gamma} \|C_i(k)\Phi(k, 1)g\|_2^2\right)^{\frac{1}{2}} \\ \text{for all } g \in \mathbb{R}^n \end{array} \right\} \tag{7.30}$$

for $\sigma = (\sigma_1, \ldots, \sigma_\mu), \sigma_i \geqslant 0$, $i = 1, \ldots, \mu$. Thus, the system (7.16) is locally null-controllable on $\{0, \gamma\}$ iff for each $\varepsilon > 0, V_\varepsilon \neq \phi$ or iff for some σ such that $\sigma_i \geqslant 0, E_\sigma \neq \phi$.

Lemma 7.1. If $V_\varepsilon \neq \phi, \varepsilon > 0,$ then $\exists\, \hat{\sigma}_\varepsilon = (\hat{\sigma}_1, \ldots, \hat{\sigma}_\mu) \in V_\varepsilon$ such that

$$\|\hat{\sigma}_\varepsilon\|_2 = \inf\{\|\sigma\|_2 : \sigma \in V_\varepsilon\} , \tag{7.31}$$

and there exists $\hat{g} \in \mathbb{R}^n$ with $\|\hat{g}\|_2 = 1$ such that

$$\varepsilon\|\hat{g}\|_2 = \sum_{i=1}^{\mu} \hat{\sigma}_i \left(\sum_{k=1}^{\gamma} \|C_i(k)\Phi(k, 1)\hat{g}\|_2^2\right)^{\frac{1}{2}} . \tag{7.32}$$

Lemma 7.2. If $E_\sigma \neq \phi, \sigma_i \geq 0$ then $\exists\, \hat{\varepsilon}_\sigma \in E_\sigma$ such that

$$\hat{\varepsilon}_\sigma = \sup\{\varepsilon > 0 : \varepsilon \in E_\sigma\} \tag{7.33}$$

and thus $\exists\, \bar{g} \in \mathbb{R}^n$ with $\|\bar{g}\|_2 = 1$ such that

$$\hat{\varepsilon}_\sigma \|\bar{g}\|_2 = \sum_{i=1}^{\mu} \sigma_i \left(\sum_{k=1}^{\gamma} \|C_i(k)\Phi(k, 1)\bar{g}\|_2^2\right)^{\frac{1}{2}} . \tag{7.34}$$

Theorem 7.6. If $V_\varepsilon \neq \phi, \varepsilon > 0$, and

$$\bar{\varepsilon} = \sup\{\delta > 0 : \delta \in E_{\hat{\sigma}_\varepsilon}\} , \tag{7.35}$$

then $\bar{\varepsilon} \in E_{\hat{\sigma}_\varepsilon}$ and $\bar{\varepsilon} = \varepsilon$.

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Chapter 2

Infinite Dimensional Linear Systems With Multiple Norm-bounded Controllers

1. Introduction

Controllability and observability of infinite dimensional linear systems have received much attention in recent years. Important contributors in these areas include Balakrishnan, Fattorini, Lions and Russell, to name just a few. For a comprehensive account, the recent books of Curtain and Pritchard [3], [4] and Balakrishnan [1] as well as the excellent survey of Russell [13] should be consulted. Most of the studies in controllability as reported in the above works are concerned with a single unconstrained controller. The present chapter consider the problem of controllability of infinite dimensional linear systems with multiple independent controllers each of which is norm-bounded. We develop new necessary and sufficient conditions for systems to be such. Our study includes both the usual (cooperative) controllability as well as the game (non-cooperative) controllability. For the former, we are motivated by the growing interest of decentralized controls [14] in finite dimensional linear systems while the latter by the works on pursuit games in [7], [8], [9] and those on linear partial differential games in [2], [10]. The problem of sending a given initial state to some target sets is examined. Applications to linear differential delay systems as reported in [5], [12] and systems described by simple linear partial differential equations are given. For the cooperative mode, we give a duality theory between controllability and observability. The novelty here is that the controls in the dual systems are also norm-bounded.

2. Basic Problem and Preliminaries

Let X and U_i, $i = 1, \ldots, \mu$ be real Banach spaces and let $x(t)$ be a function from $\mathbb{R}^+$ into X, satisfying the evolution equation

$$\frac{dx}{dt} + A(t)x = \sum_{i=1}^{\mu} B_i(t)u_i(t) \tag{2.1}$$

with the initial condition

$$x(0) = x_0 \tag{2.2}$$

where $A(t)$ is a linear operator on X, unbounded in general, and $B_i(t)$ is a bounded linear operator from a reflexive Banach space U_i to the Banach space X, and is strongly continuous in t. Furthermore, we need the following assumptions [6] :

(i) $A(t)$ is a closed operator in X with a dense domain $D(A)$ which is independent of t.

(ii) For each $t \in \mathbb{R}^+$, the resolvent $R(\lambda; A(t)) = [\lambda I - A(t)]^{-1}$ exists for all λ with $\mathrm{Re}\,\lambda \leq 0$ and

$$\|R(\lambda; A(t))\| \leq \frac{C}{1 + |\lambda|}, \quad \mathrm{Re}\,\lambda \leq 0 \tag{2.3}$$

(iii) For any t, τ, s in $\mathbb{R}^+$

$$\|[A(t) - A(\tau)]A^{-1}(s)\| \leq C\,|t - \tau|^{\alpha}, \quad 0 < \alpha < 1 \tag{2.4}$$

where the constants C, α are independent of t, τ, s, i.e. the bounded operator $A(t)A^{-1}(s)$ is Hölder continuous in t in the uniform operator topology for each fixed s.

Then, there exists a unique fundamental solution $S(t, \tau)$ of (2.1), belonging to $B(X)$; being strongly continuous in $t, \tau \in [0, \infty)$ and

(a) the derivative $\frac{\partial}{\partial t} S(t, \tau)$ exists in the strong topology and belongs to $B(X)$ for $0 \leq \tau < t < \infty$ and it is also strongly continuous in t, $0 \leq \tau < t < \infty$;

(b) the range $R(S(t, \tau))$ is in $D(A)$;

(c) $\frac{\partial}{\partial t} S(t, \tau) + A(t)S(t, \tau) = 0 , \quad 0 \leq \tau < t < \infty$

$$S(\tau, \tau) = I , \qquad 0 \leq \tau < \infty .$$

Let the controls $u_i(t)$, $i = 1, \ldots, \mu$ be in the abstract Lebesgue locally integrable spaces which will be specified later. We admit a mild solution

$$x(t) = S(t, 0)x_o + \sum_{i=1}^{\mu} \int_0^t S(t, \tau)B_i(\tau)u_i(\tau)d\tau \tag{2.5}$$

where the integration is in Bochner sense.

If $A(t) = A$, $B_i(t) = B_i$, $i = 1, \ldots, \mu$ are independent of the time t, then the fundamental solution of (2.1) has the form $S(t - \tau)$. We see that, assumption (ii) implies

(ii)' $\lambda \in \rho(A)$, the resolvent set, for $\operatorname{Re} \lambda \leq 0$ and

$$\| (R(\lambda; A))^n \| \leq \frac{C}{|\lambda|^n} .$$

If (i) and (ii)' holds, then $-A$ generates a strongly continuous semigroup $\{S(t)\}$, $t \in \mathbb{R}^+$ and have the following properties:

(1) $S(t)$ can be continued analytically in $\mathbb{R}^+$;

(2) $AS(t), \frac{d}{dt} S(t)$ are bounded operators, $t \in \mathbb{R}^+$ and

$$\frac{d}{dt} S(t)x = -AS(t)x , \qquad x \in X .$$

We denote $S(t) = \exp(-tA)$.

Let

$$L_{p_i}(0, T; U_i) = \{u_i : [0, T] \to U_i \text{ is abstract Lebesgue measurable and } \int_0^T \|u_i(t)\|_{U_i}^{p_i} dt < \infty \} \tag{2.6}$$

and also denote the equivalence classes by $L_{p_i}(0, T; U_i)$ with respect to the norm

$$|||u_i|||_{p_i} = (\int_0^T \|u_i(t)\|_{U_i}^{p_i} dt)^{1/p_i} , \quad 1 \leq p_i < \infty , \tag{2.7}$$

then $L_{p_i}(0, T; U_i)$, $1 \leq p_i < \infty$, are Banach spaces. Since we assume U_i, $i = 1, \ldots, \mu$ are reflexive, then $L_{p_i}(0, T; U_i)$, $1 < p_i < \infty$, are reflexive in the sense that

$$L^*_{p_i}(0, T; U_i) \cong L_{q_i}(0, T; U^*_i)$$

where $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = 1, \ldots, \mu$.

From now on, we assume $1 < p_i < \infty$ and $\rho_i > 0$, $i = 1, \ldots, \mu$, and let

$$L^{\rho_i}_{p_i}(0, T; U_i) = \{u_i \in L_{p_i}(0, T; U_i) : |||u_i|||_{p_i} \leq \rho_i\} \tag{2.8}$$

$$L^{\rho_i}_{p_i}(U_i) = \bigcup_{T \geq 0} L^{\rho_i}_{p_i}(0, T; U_i) \tag{2.9}$$

be the constrained sets for u_i , $i = 1, \ldots, \mu$.

Lemma 2.1. The sets $L_{p_i}^{\rho_i}(0, T; U_i)$, $i = 1, \ldots, \mu$, are convex and weakly compact in $L_{p_i}(0, T; U_i)$ respectively.

Proof: The convexity, boundedness and closedness of $L_{p_i}^{\rho_i}(0, T; U_i)$ are obvious. Hence, they are weakly closed and bounded in the reflexive Banach spaces $L_{p_i}(0, T; U_i)$, so that the weak compactness of $L_{p_i}^{\rho_i}(0, T; U_i)$, $1 < p_i < \infty$, follows.

Lemma 2.2. For each $T \in \mathbb{R}^+$, the mappings

$$\Lambda_i : L_{p_i}(0, T; U_i) \to X$$

given by

$$\Lambda_i(u_i) = \int_0^T S(T, \tau)B_i(\tau)u_i(\tau)d\tau \ , \qquad i = 1, \ldots, \mu \ ,$$

are linear continuous operators and $\Lambda_i(L_{p_i}^{\rho_i}(0, T; U_i))$, $i = 1, \ldots, \mu$, are convex and weakly compact.

Proof: Linearity of Λ_i is obvious for each $T \in \mathbb{R}^+$ and

$$\begin{aligned}\|\Lambda_i(u_i)\|_X &\leq \int_0^T \|S(T, \tau)B_i(\tau)u_i(\tau)\|_X d\tau \\ &\leq \int_0^T \|S(T, \tau)B_i(\tau)\|_{L(U_i, X)} \|u_i(\tau)\|_{U_i} d\tau \\ &\leq \left(\int_0^T \|S(T, \tau)B_i(\tau)\|_{L(U_i, X)}^{q_i} d\tau\right)^{1/q_i} \left(\int_0^T \|u_i(\tau)\|_{U_i}^{p_i} d\tau\right)^{1/p_i} .\end{aligned}$$

Since $S(T, \tau)$, $B_i(\tau)$ are bounded operators and are strongly continuous in τ, then

$$\|\Lambda_i\|_{L(L_{p_i}(0, T; U_i), X)} \leq \left(\int_0^T \|S(T, \tau)B_i(\tau)\|_{L(U_i, X)}^{q_i} d\tau\right)^{1/q_i}$$

so that $\wedge_i$ is bounded. Thus, $\wedge_i$ is continuous with respect to the weak topologies in $L_{p_i}(0, T; X)$ and X, so that the sets

$$R_i(T) = \wedge_i(L_{p_i}^{\rho_i}(0, T; U_i)) , \quad i = 1, \ldots, \mu \tag{2.10}$$

are convex and weakly compact in X since $L_{p_i}^{\rho_i}(0, T; U_i)$ is weakly compact by lemma 2.1.

Lemma 2.3. If E and F are closed convex sets in a Banach space X with E being weakly compact, then $E + F$ is closed convex in X. In addition, if F is weakly compact, then $E + F$ is weakly compact in X.

Proof: Convexity of $E + F$ is obvious. Let $g \in \overline{E + F}$ in X_w with weak topology. Then, $\exists \{e_n\} \subset E$, $\{f_n\} \subset F$ such that $e_n + f_n \to g$ weakly, i.e. $x^*(e_n + f_n) = x^*(e_n) + x^*(f_n) \to x^*(g)$, $\forall x^* \in X_w^* = X^*$. Since E is weakly compact, then $\exists$ subsequent $\{e_{n_k}\} \subset \{e_n\}$ s.t. $x^*(e_{n_k}) \to x^*(e_o)$, $e_o \in E$ $\forall x^* \in X^*$. Then, $x^*(f_{n_k}) \to x^*(g) - x^*(e_o) = x^*(g - e_o)$, $\forall x^* \in X^*$. As F is closed convex, F is weakly closed and so $g - e_o \in F$. Hence $E + F$ is weakly closed and so closed. If F is also weakly compact, then $E \times F$ is compact in the product space $X_w \times X_w$. Let $\{a_n\}$, $\{b_n\}$ be sequences in X_w such that $a_n \to a_o$, $b_n \to b_o$ in X_w. Then,
$x^*(a_n + b_n) = x^*(a_n) + x^*(b_n) \to x^*(a_o) + x^*(b_o) = x^*(a_o + b_o)$, $\forall x^* \in X^*$
which shows that the addition operator $+ : X_w \times X_w \to X_w$ is continuous. Therefore, $+(E \times F) = E + F$ is weakly compact.

Lemma 2.4. If E and F are non-empty closed convex sets in a Banach space X with one of them being weakly compact, then

(i) $E \cap F \neq \phi$ iff for all $g \in X^*$,

$$\inf_{e \in E} \langle g, e\rangle \leq \sup_{f \in F} \langle g, f\rangle \quad ; \tag{2.11}$$

(ii) $E \subseteq F$ iff for all $g \in X^*$,

$$\sup_{e \in E} \langle g, e\rangle \leq \sup_{f \in F} \langle g, f\rangle \ . \tag{2.12}$$

Proof: (i) Necessity is obvious. Since E and F are weakly closed and convex sets in a locally convex space X_w with weak topology and either E or F is compact in X_w . If $E \cap F = \phi$, then by the strict separation theorem, there exists $0 \neq \hat{g} \in X_w^* = X^*$ such that

$$\sup_{f \in F} \langle \hat{g}, f\rangle < \inf_{e \in E} \langle \hat{g}, e\rangle \ .$$

Thus, (2.11) cannot hold for all $g \in X^*$.

(ii) Necessity is obvious. If $h \in E \backslash F$, then $\{h\}$ is compact and convex in X . Thus, by the strict separation theorem, there exists $0 \neq \bar{g} \in X^*$ such that

$$\sup_{f \in F} \langle \bar{g}, f\rangle < \langle \bar{g}, h\rangle \leq \sup_{e \in E} \langle \bar{g}, e\rangle \ .$$

Hence, (2.12) cannot hold for all $g \in X^*$.

3. Controllability.

We are in the position to investigate the controllability of the system (2.1) with input constraints (2.8), (2.9).

Definition 3.1. The system (2.1) is said to be null-controllable at x_o on $[0, T]$ if $\exists\, u_i \in L_{p_i}^{\rho_i}(0, T; U_i)$, $i = 1, \ldots, \mu$, such that x_o can be steered to the origin.

Definition 3.2. The system (2.1) is said to be x_1-controllable at x_o if $\exists\, T \in \mathbb{R}^+$ and $u_i \in L_{p_i}^{\rho_i}(0, T; U_i)$, $i = 1, \ldots, \mu$, such that x_1 can be reached from x_o .

Remark 3.1. Since $R(S(t, \tau)) \subseteq D(A)$, then if $x_1 \notin D(A)$, we see that x_1 may not be reached from x_o .

Definition 3.3. The system (2.1) is said to be approximately controllable at x_o on $[0, T]$ if for each $x_1 \in X$ and $\varepsilon > 0$, $\exists\, \hat{x}_1 \in X$ with $\|x_1 - \hat{x}_1\|_X \leq \varepsilon$ such that $\hat{x}_1$ can be reached from x_o at time T .

The main results will now be presented in the following theorems.

Theorem 3.1. The system (2.1) is x_1-controllable at x_o iff $\exists\, T \geq 0$ and for all $g \in X^*$,

$$|\langle g, x_1 - S(T, 0)x_o \rangle| \leq \sum_{i=1}^{\mu} \rho_i \left(\int_0^T \|B_i^*(\tau) S^*(T, \tau) g\|_{U_i^*}^{q_i} \, d\tau \right)^{1/q_i} \tag{3.1}$$

where $(\cdot)^*$ denotes the adjoint of the respective operator.

Proof: By definition 3.2, the system (2.1) is x_1-controllable at x_o iff $\exists\, T \geq 0$ such that

$$x_1 - S(T, 0)x_o \in \sum_{i=1}^{\mu} R_i(T) \qquad (3.2)$$

which is convex and weakly compact in X by lemma 2.2 and lemma 2.3. Thus, by lemma 2.4, (3.2) holds iff for all $g \in X^*$,

$$\begin{aligned}
\langle g, x_1 - S(T, 0)x_o\rangle &\leq \sup_{\substack{u_i \in L_{p_i}^{\rho_i}(0,T;\, U_i)\\ i = 1, \ldots, \mu}} \langle g, \sum_{i=1}^{\mu} \Lambda_i(u_i)\rangle \\
&= \sum_{i=1}^{\mu} \sup_{u_i \in L_{p_i}^{\rho_i}(0,\, T;\, U_i)} \langle g, \int_o^T S(T, \tau)B_i(\tau)u_i(\tau)d\tau\rangle \\
&= \sum_{i=1}^{\mu} \sup_{u_i \in L_{p_i}^{\rho_i}(0,\, T;\, U_i)} \int_o^T \langle B_i^*(\tau)S^*(T, \tau)g, u_i(\tau)\rangle d\tau \\
&= \sum_{i=1}^{\mu} \rho_i \left(\int_o^T \|B_i^*(\tau)S^*(T, \tau)g\|_{U_i^*}^{q_i} d\tau\right)^{1/q_i} . \qquad (3.3)
\end{aligned}$$

Replacing g by $-g$, we have (3.1).

Definition 3.4. The system (2.1) is said to be locally x_1-controllable at x_o on $[0, T]$ if there exists a neighborhood $N(x_o)$ of x_o in X such that for each $y_o \in N(x_o)$, it is x_1-controllable at y_o on $[0, T]$.

Remark 3.2. $N(x_o)$ can be taken as the closed ball $B(x_o, \varepsilon) = \{x \in X : \|x - x_o\|_X \leq \varepsilon\}$.

Theorem 3.2. The system (2.1) is locally x_1-controllable at x_o on $[0, T]$ iff $\exists\, \varepsilon > 0$ and for all $g \in X^*$,

$$|\langle g, x_1 - S(T, 0)x_o\rangle| + \varepsilon\|S^*(T, 0)g\|_{X^*} \leq \sum_{i=1}^{\mu} \rho_i \left(\int_0^T \|B_i^*(\tau)S^*(T, \tau)g\|_{U_i^*}^{q_i} d\tau\right)^{1/q_i}. \quad (3.4)$$

Proof: In view of (3.3) and definition 3.4, the system (2.1) is locally x_1-controllable at x_o on $[0, T]$ iff $\exists\, \varepsilon > 0$ and for all $g \in X^*$,

$$\begin{aligned}
&\sup_{y_o \in B(x_o,\varepsilon)} \langle g, x_1 - S(T, 0)y_o\rangle \\
&= \langle g, x_1 - S(T, 0)x_o\rangle + \sup_{y_o \in B(0,\varepsilon)} \langle g, S(T, 0)y_o\rangle \\
&= \langle g, x_1 - S(T, 0)x_o\rangle + \varepsilon\|S^*(T, 0)g\|_{X^*}
\end{aligned}$$

is less than or equal to the right hand side of (3.1). Replacing g by $-g$, we have (3.4).

Theorem 3.3. The system (2.1) is approximately controllable at x_o on $[0, T]$ iff for each $x_1 \in X$ and any $\varepsilon > 0$,

$$|\langle g, x_1 - S(T, 0)x_o\rangle| \leq \varepsilon\|g\|_{X^*} + \sum_{i=1}^{\mu} \rho_i \left(\int_0^T \|B_i^*(\tau)S^*(T, \tau)g\|_{U_i^*}^{q_i} d\tau\right)^{1/q_i} \quad (3.5)$$

for all $g \in X^*$.

Proof: By definition 3.3, the system (2.1) is approximately controllable at x_o on $[0, T]$ iff for each $x_1 \in X$ and any $\varepsilon > 0$, $\exists\, \hat{x}_1 \in B(x_1, \varepsilon)$ such that (3.2) holds with $x_1 = \hat{x}_1$ or

$$B(x_1, \varepsilon) - \{S(T, 0)x_o\} \cap \sum_{i=1}^{\mu} R_i(T) \neq \phi .$$

By lemma 2.4(i), we have for all $g \in X^*$,

$$\inf_{y \in B(x_1,\, \varepsilon)} \langle g,\, y - S(T,\, 0)x_o \rangle \leq \sup_{\substack{u_i \in L^{\rho_i}_{p_i}(0,T;\, U_i) \\ i=1,\, \ldots,\, \mu}} \langle g,\, \sum_{i=1}^{\mu} \Lambda_i(u_i) \rangle \,. \tag{3.6}$$

The left hand side of (3.6) becomes

$$\langle g,\, x_1 - S(T,\, 0)x_o \rangle - \sup_{y \in B(0,\, \varepsilon)} \langle g,\, -y \rangle$$

$$= \langle g,\, x_1 - S(T,\, 0)x_o \rangle - \varepsilon \|g\|_{X^*}$$

while the right hand side of (3.6) equals to

$$\sum_{i=1}^{\mu} \rho_i \left(\int_0^T \|B_i^*(\tau)\, S^*(T,\, \tau) g\|^{q_i}_{U_i^*} \, d\tau \right)^{1/q_i} .$$

Thus, replacing g by $-g$ in (3.6), we have (3.5).

4. Game Problems and Target sets

Now, we consider a game problem in a real Banach space X with two non-cooperative teams. One represents a pursuer whose controls u_i, $i = 1, \ldots, \mu$, are in the corresponding reflexive real Banach spaces $L_{p_i}(0, T; U_i)$ with U_i being reflexive Banach spaces; while the other one represents a evader whose controls v_j, $j = 1, \ldots, \nu$, are in the corresponding reflexive real Banach spaces $L_{p'_j}(0, T; V_j)$ with V_j being reflexive real Banach spaces. The game system is described by the following evolution equation

$$\frac{dx}{dt} + A(t)x(t) = \sum_{i=1}^{\mu} B_p^i(t)u_i(t) + \sum_{j=1}^{\nu} B_e^j(t)v_j(t) \tag{4.1}$$

where $B_p^i(t)$, $B_e^j(t)$ are bounded operators and are strongly continuous in $t \in \mathbb{R}^+$. The assumptions on $A(t)$ are the same as in section 2 and the mild solution is given by

$$x(t) = S(t, 0)x_o + \sum_{i=1}^{\mu} \int_o^t S(t, \tau)B_p^i(\tau)u_i(\tau)d\tau$$
$$+ \sum_{j=1}^{\nu} \int_o^t S(t, \tau)B_e^j(\tau)v_j(\tau)d\tau .$$

The constraints on u_i are as before, while constraints for the evader's controls are

$$L_{p'_j}^{\sigma_j}(0, T; V_j) = \{v_j \in L_{p'_j}(0, T; V_j) : |||v_j|||_{p'_j} \leq \sigma_j\} , \tag{4.2}$$

$$L_{p'_j}^{\sigma_j}(V_j) = \bigcup_{T \geq 0} L_{p'_j}^{\sigma_j}(0, T; V_j) , \tag{4.3}$$

where $\sigma_j > 0$, and the reachable sets

$$R_e^j(T) = \wedge_e^j(L_{p_j'}^{\sigma_j}(0, T; V_j)) , \tag{4.4}$$

$$R_p^i(T) = \wedge_p^i(L_{p_i}^{\rho_i}(0, T; U_i)) \tag{4.5}$$

are weakly compact and convex in X , where $\frac{1}{p_j'} + \frac{1}{q_j'} = 1$, $1 < p_j' < \infty$ and

$$\wedge_e^j(v_j) = \int_0^T S(T, \tau)B_e^j(\tau)v_j(\tau)d\tau , \tag{4.6}$$

$$\wedge_p^i(u_i) = \int_0^T S(T, \tau)B_p^i(\tau)u_i(\tau)d\tau , \tag{4.7}$$

$j = 1, \ldots, \nu$; $i = 1, \ldots, \mu$.

Definition 4.1. The game system (4.1) is said to be null-controllable at x_o on $[0, T]$ if for each evader's controls $v_j \in L_{p_j'}^{\sigma_j}(0, T; V_j)$, $j = 1, \ldots, \nu$; there exists pursuer's controls $u_i \in L_{p_i}^{\rho_i}(0, T; U_i)$, $i = 1, \ldots, \mu$, such that $x(T) = 0$.

Theorem 4.1. The game system (4.1) is null-controllable at x_o on $[0, T]$ iff for all $g \in X^*$,

$$\begin{aligned} |\langle g, S(T, 0)x_o\rangle| \leq & \sum_{i=1}^{\mu} \rho_i \Big(\int_0^T ||B_p^{i*}(\tau)S^*(T, \tau)g||_{U_i^*}^{q_i} d\tau\Big)^{1/q_i} \\ & - \sum_{j=1}^{\nu} \sigma_j \Big(\int_0^T ||B_e^{j*}(\tau)S^*(T, \tau)g||_{V_j^*}^{q_j'} d\tau\Big)^{1/q_j'} . \end{aligned} \tag{4.8}$$

Proof: By definition 4.1, the game system (4.1) is null-controllable at x_o on $[0, T]$ iff

$$S(T, 0)x_o + \sum_{j=1}^{\nu} R_e^j(T) \subseteq - \sum_{i=1}^{\mu} R_p^i(T) = \sum_{i=1}^{\mu} R_p^i(T) \tag{4.9}$$

by the symmetry of $R_p^i(T)$. Since $\sum_{j=1}^{\nu} R_e^j(T)$ and $\sum_{i=1}^{\mu} R_p^i(T)$ are convex and weakly compact in X by lemma 2.3, thus, in view of lemma 2.4(ii), (4.9) holds iff

$$\sup_{\substack{v_j \in L_{p_j'}^{\sigma_j}(0,\, T;\, V_j)\\ j=1,\,\ldots,\,\nu}} \langle g,\ S(T,\ 0)x_o + \sum_{j=1}^{\nu} \Lambda_e^j(v_j)\rangle \leq \sup_{\substack{u_i \in L_{p_i}^{\rho_i}(0,T;\, U_i)\\ i=1,\,\ldots,\,\mu}} \langle g,\ \sum_{i=1}^{\mu} \Lambda_p^i(u_i)\rangle$$

holds iff

$$\langle g,\ S(T,\ 0)x_o\rangle + \sum_{j=1}^{\nu} \sigma_j \left(\int_o^T \|B_e^{j*}(\tau)S^*(T,\ \tau)g\|_{V_j^*}^{q_j'} d\tau\right)^{1/q_j'}$$
$$\leq \sum_{i=1}^{\mu} \rho_i \left(\int_o^T \|B_p^{i*}(\tau)S^*(T,\ \tau)g\|_{U_i^*}^{q_i} d\tau\right)^{1/q_i} \tag{4.10}$$

holds. Hence, replacing g by $-g$ in (4.10), we have (4.8).

In many cases of interest, we only want the final states to be in some given target set Ω in X , so we need the following definition.

Definition 4.2. The game system (4.1) is said to be Ω-controllable at x_o on [0, T] if for each evader's controls, there exists pursuer's controls such that $x(T) \in \Omega$.

Theorem 4.2. If Ω is closed convex in X , then the game system (4.1) is Ω-controllable at x_o on [0, T] iff for all $g \in X^*$,

$$\langle g,\ S(T,\ 0)x_o\rangle \leq \sup_{y\in\Omega} \langle g,\ y\rangle + \sum_{i=1}^{\mu} \rho_i \left(\int_o^T \|B_p^{i*}(\tau)S^*(T,\ \tau)g\|_{U_i^*}^{q_i} d\tau\right)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \left(\int_o^T \|B_e^{j*}(\tau)S^*(T,\ \tau)g\|_{V_j^*}^{q_j'} d\tau\right)^{1/q_j'} .$$

We omit the proof of theorem 4.2 as it is similar to that of theorem 4.1.

Remark 4.1. (i) If $\Omega = \{0\}$, then theorem 4.2 reduces to theorem 4.1

(ii) If $\Omega = B(y_0, r)$, then

$$\sup_{y \in B(y_0, r)} \langle g, y \rangle = \langle g, y_0 \rangle + \sup_{y \in B(0, r)} \langle g, y \rangle$$
$$= \langle g, y_0 \rangle + r\|g\|_{X^*} .$$

If Ω is a null subspace of a continuous projection $P : X \to X$, then $X = R(P) \oplus N(P)$ where the range $R(P)$ and the kernel $N(P) = \Omega$ are closed subspaces. Thus, $x(T) \in \Omega$ iff $P(x(T)) = 0$. Hence, we have the following theorem which is similar to theorem 4.1.

Theorem 4.3. If $\Omega = N(P)$, then the game system (4.1) is Ω-controllable at x_0 on $[0, T]$ iff for all $g \in X^*$,

$$|\langle g, PS(T, 0)x_0 \rangle| \leq \sum_{i=1}^{\mu} \rho_i \left(\int_0^T \|B_p^{i*}(\tau) S^*(T, \tau) P^* g\|_{U_i^*}^{q_i} d\tau \right)^{1/q_i}$$
$$- \sum_{j=1}^{\nu} \sigma_j \left(\int_0^T \|B_e^{j*}(\tau) S^*(T, \tau) P^* g\|_{V_j^*}^{q'_j} d\tau \right)^{1/q'_j} .$$

Remark 4.2. If Ω is a closed subspace of a Hilbert space X, then a continuous projection P exists such that $N(P) = \Omega$ and $X = R(P) \oplus \Omega$.

5. Applications

There are many applications of differential equations defined in infinite dimensional abstract spaces especially in the cases of partial differential equations and functional differential equations. We give several simple examples to illustrate the applicability of our theory developed in the previous sections.

Example 5.1. Retarded functional differential equation [3], [5], [12]. The class of systems under study is described by

$$\begin{cases} \frac{dx}{dt} = A_1 x(t) + A_2 x(t-r) + \int_{-r}^{0} A_3(\theta) x(t+\theta) d\theta + Bu(t) \\ x(t) = h(t) , \quad -r \leq t \leq 0 \end{cases} \tag{5.1}$$

where $A_1, A_2, B \in B(H)$ bounded linear functions on H , H is a Hilbert space and $A_3 \in C(-r, 0; B(H))$. Let $M_2(-r, 0; H)$ be the space of equivalence classes of functions in $L_2(-r, 0; H)$ under the equivalence relation

$$\langle y, z \rangle_{M_2} = \langle y(0), z(0) \rangle_H + \int_{-r}^{0} \langle y(t), z(t) \rangle_H \, dt$$

and is isometrically isomorphic to the Hilbert space $H \times L_2(-r, 0; H)$. Then (5.1) is equivalent to the following abstract control problem on $M_2(-r, 0; H)$

$$\begin{cases} \frac{dy}{dt} = Ay(t) + \overline{B}u(t) \\ y(0) = h \end{cases} \tag{5.2}$$

where

$$(\overline{B}u)(\theta) = \begin{cases} Bu , & \theta = 0 \\ 0 , & \theta \in [-r, 0) \end{cases} ;$$

$$(Ah)(\theta) = \begin{cases} A_1 h(0) + A_2 h(-r) + \int_{-r}^{0} A_3(\alpha) h(\alpha) d\alpha, & \theta = 0 \\ \dfrac{dh}{d\theta} & , \quad \theta \in [-r, 0) \end{cases}$$

then $\bar{B} \in B(H, M_2)$ and A is a closed linear operator on M_2 with domain

$$D(A) = W_2^{(1)}(-r, 0; H) .$$

Define

$$x_t(\theta) = x(t + \theta) , \qquad \theta \in [-r, 0]$$

where $x(t)$ is a homogeneous solution of (5.1). Thus, A is the infinitesimal generator of the strongly continuous semigroup $\{S(t)\}$, $t \geq 0$ on M_2 defined by

$$(S(t)h)(\theta) = \begin{cases} x(t) & , \quad \theta = 0 \\ x_t(\theta) & , \quad \theta \in [-r, 0) \end{cases} .$$

Let $u \in L_2^{\rho}(H)$ be locally square integrable constrained set. The mild solution is given by

$$y(t) = S(t)h + \int_0^t S(t - \tau)\bar{B}u(\tau) d\tau .$$

Thus, the system (5.2) is null-controllable at h iff $\exists\, T \geq 0$ and for all $g \in M_2^* = M_2$,

$$|\langle g, S(T)h\rangle_{M_2}| \leq \rho \left(\int_0^T \|\bar{B}^* S^*(T - \tau) g\|_H^2 d\tau \right)^{\frac{1}{2}} .$$

Let

$$P_H \; : \; M_2(-r, 0; H) \to H$$

$$P_{L_2} \; : \; M_2(-r, 0; H) \to L_2(-r, 0; H)$$

be projections on M_2 such that

$$R(P_H) = N(P_{L_2}) = H\ , \qquad R(P_{L_2}) = N(P_H) = L_2(-r,\ 0;\ H)\ .$$

Therefore, the system (5.2) is H-controllable at x_o with target set H iff $\exists\ T \geq 0$ and for all $g \in L_2(-r,\ 0;\ H)$,

$$|\langle g,\ P_{L_2} S(T)h\rangle_{L_2}| \leq \rho\left(\int_0^T \|\overline{B}^* S^*(T-\tau)P_{L_2}^* g\|_H^2\, d\tau\right)^{\frac{1}{2}}\ ,$$

while the system (5.2) is L_2-controllable at x_o with target set $L_2(-r,\ 0;\ H)$ iff $\exists\ T \geq 0$ and for all $g \in H$,

$$|\langle g,\ P_H S(T)h\rangle_H| \leq \rho\left(\int_0^T \|\overline{B}^* S^*(T-\tau)P_H^* g\|_H^2\, d\tau\right)^{\frac{1}{2}}\ .$$

Example 5.2. Heat equation

Let us consider

$$x_t = x_{\zeta\zeta} + u + v\ , \quad 0 < \zeta < \pi\ , \quad t > 0 \tag{5.3}$$

with boundary conditions

$$x(0,\ t) = 0,\ x(\pi,\ t) = 0\ , \qquad t > 0 \tag{5.4}$$

and initial condition

$$x(\zeta,\ 0) = f(\zeta)\ , \qquad 0 < \zeta < \pi\ ; \tag{5.5}$$

where $x(\cdot,\ t) \in X = L_2[0,\ \pi]$,

$$u \in L_2^{\rho}(X)\ , \qquad v \in L_2^{\sigma}(X)\ ; \qquad \rho > \sigma > 0\ .$$

By the method of separation of variables, the solution of homogeneous equation of (5.3) satisfying (5.4), (5.5) in Fourier expansion is

$$x(\zeta, t) = \sum_{n=1}^{\infty} \hat{f}(n) e^{-n^2 t} \sin n\zeta$$

where

$$\hat{f}(n) = \frac{2}{\pi} \int_0^{\pi} f(\zeta) \sin n\zeta \, d\zeta \quad .$$

Define

$$S(t) f(\zeta) = \sum_{n=1}^{\infty} \hat{f}(n) e^{-n^2 t} \sin n\zeta \ ,$$

then $\{S(t)\}$, $t \geq 0$ is a strongly continuous semigroup with

$$\|S(t)\|_{L(X)} \leq e^{-t} \quad .$$

The infinitesimal generator

$$Ax = -x_{\zeta\zeta}$$

with dense domain

$$D(A) = H^2[0, \pi] \cap H_0^1[0, \pi]$$

in X . Since A is self-adjoint, so is $S(t)$. As $\{\sin n\zeta\}_{n=1}^{\infty}$ is a complete orthogonal base for $L_2[0, \pi]$, then for any $g \in L_2[0, \pi]$,

$$g(\zeta) = \sum_{n=1}^{\infty} \hat{g}(n) \sin n\zeta \quad .$$

In order to investigate the controllability of system (5.3), we compute the following terms:

$$\int_0^T \|S^*(T - \tau) g\|_X^2 \, d\tau$$

$$= \int_0^T \|\sum_{n=1}^{\infty} \hat{g}(n) e^{-n^2 (T-\tau)} \sin n\zeta \|_X^2 \, d\tau$$

$$= \sum_{n=1}^{\infty} \frac{\pi}{2} \hat{g}^2(n) \int_0^T e^{-2n^2(T-\tau)} d\tau$$

$$= \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\hat{g}^2(n)}{n^2} (1 - e^{-2n^2 T}) \tag{5.6}$$

and

$$\langle g, S(T)f \rangle_X = \langle \sum_{n=1}^{\infty} \hat{g}(n) \sin n\zeta, \sum_{k=1}^{\infty} \hat{f}(k) e^{-k^2 T} \sin k\zeta \rangle_X = \frac{\pi}{2} \sum_{n=1}^{\infty} \hat{g}(n)\hat{f}(n) e^{-n^2 T}$$

$$\leq \frac{\pi}{2} \left(\sum_{n=1}^{\infty} \frac{\hat{g}^2(n)}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^2 \hat{f}^2(n) e^{-2n^2 T}\right)^{\frac{1}{2}} .$$

Since $f \in L_2[0, \pi]$, we have $|\hat{f}(n)| \leq K$, $\forall\, n$, for some $K > 0$. Thus,

$$\sum_{n=1}^{\infty} n^2 \hat{f}^2(n) e^{-2n^2 T} \leq K^2 \sum_{n=1}^{\infty} \frac{n^2}{e^{2n^2 T}}$$

$$= K^2 \sum_{n=1}^{\infty} \frac{n^2}{1 + 2n^2 T + \frac{4n^4 T^2}{2!} + \dots}$$

$$\leq K^2 \sum_{n=1}^{\infty} \frac{n^2}{2n^4 T^2}$$

$$= \frac{K^2}{2T^2} \left(\frac{\pi^2}{6}\right) = \frac{\pi^2 K^2}{12 T^2} .$$

$$\therefore \quad |\langle g, S(T)f \rangle_X| \leq \frac{\pi^2 K}{4\sqrt{3}\, T} \left(\sum_{n=1}^{\infty} \frac{\hat{g}^2(n)}{n^2}\right)^{\frac{1}{2}} . \tag{5.7}$$

But in view of (5.6),

$$(\rho - \sigma)\left(\int_0^T \|S^*(T - \tau)g\|_X^2 d\tau\right)^{\frac{1}{2}} \geq (\rho - \sigma)\frac{\sqrt{\pi}}{2}(1 - e^{-2T})^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{\hat{g}^2(n)}{n^2}\right)^{\frac{1}{2}} . \tag{5.8}$$

Comparing (5.7) and (5.8), we see that for sufficiently large T , (5.7) $\leq$ (5.8) for all $g \in X$. In order to find the time T , depending on the given initial state f , let $T \geq 1$, then

$$(1 - e^{-2T})^{\frac{1}{2}} \geq (1 - \frac{1}{4})^{\frac{1}{2}} = \frac{\sqrt{3}}{2} .$$

Solving for T ,

$$\frac{\pi^2 K}{4\sqrt{3}\, T} \leq \frac{\sqrt{\pi}}{2} \frac{\sqrt{3}}{2} (\rho - \sigma) ,$$

or

$$T \geq \frac{\pi^{3/2} K}{3(\rho - \sigma)} .$$

Thus if we take

$$T = \max\{1, \frac{\pi . K}{\rho - \sigma}\} ,$$

then the system (5.3) is null-controllable at any $f \in X$.

Example 5.3 One dimensional wave equation.

Consider

$$x_{tt} - c^2 x_{\zeta\zeta} = u + v , \qquad c > 0 \tag{5.9}$$

with boundary conditions

$$x(0, t) = x(\pi, t) = 0 , \qquad 0 \leq t < \infty \tag{5.10}$$

and initial conditions

$$x(\zeta , 0) = f(\zeta), \quad x_t(\zeta , 0) = g(\zeta), \quad 0 \leq \zeta \leq \pi \quad ; \tag{5.11}$$

where $x(\cdot, t) \in X = L_2[0, \pi]$; $u \in L_2^{\rho}(X)$, $v \in L_2^{\sigma}(X)$, $\rho > \sigma > 0$. The partial derivative may be considered in the distributional sense. By the method of separation of variables,

$$x(\zeta, t) = \sum_{n=1}^{\infty} \sin n\zeta[\frac{1}{nc} \hat{g}(n)\sin n c t + \hat{f}(n)\cos n c t] .$$

The wave equation (5.9) is equivalent to the following evolution equation

$$\frac{\partial}{\partial t}\begin{pmatrix} x \\ x_t \end{pmatrix} + \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix}\begin{pmatrix} x \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} 0 \\ I \end{pmatrix} v \tag{5.12}$$

where

$$Ax = -c^2 x_{\zeta\zeta}$$

with dense domain

$$D(A) = H^2[0, \pi] \cap H_o^1[0, \pi]$$

in X. Then A is self-adjoint. But $\tilde{A}$ is not, where

$$\tilde{A} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix}, \quad \tilde{A}^* = \begin{pmatrix} 0 & A \\ -I & 0 \end{pmatrix}$$

and

$$D(\tilde{A}) = (H^2[0, \pi] \cap H_o^1[0, \pi]) \times L_2[0, \pi] .$$

Thus, $-\tilde{A}$ is an infinitesimal generator of a strongly continuous semigroup $\{S(t)\}$, $t \geq 0$, given by

$$S(t)\begin{pmatrix} f(\zeta) \\ g(\zeta) \end{pmatrix} = \begin{pmatrix} x(\zeta, t) \\ x_t(\zeta, t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} \sin n\zeta[\frac{1}{nc} \hat{g}(n) \sin nc\,t + \hat{f}(n) \cos nc\,t] \\ \sum_{n=1}^{\infty} \sin n\zeta[\hat{g}(n) \cos nc\,t - nc\,\hat{f}(n) \sin nc\,t] \end{pmatrix}$$

and

$$S^*(t)\begin{pmatrix} f(\zeta) \\ g(\zeta) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} \sin n\zeta[\hat{f}(n) \cos nc\,t - nc\,\hat{g}(n) \sin nc\,t] \\ \sum_{n=1}^{\infty} \sin n\zeta[\frac{1}{nc} \hat{f}(n) \sin nc\,t + \hat{g}(n) \cos nc\,t] \end{pmatrix},$$

so that $S(t)$ is not self-adjoint.

Let

$$B_p = B_e = \begin{bmatrix} 0 \\ -I \end{bmatrix}, \quad P = [I \quad 0] .$$

As we only want the final state $x(\zeta, T) = 0$, $\zeta \in [0, \pi]$, the target set $\Omega = N(P)$. Thus, the system (5.9) with (5.10) and (5.11) is null-controllable at $\binom{f}{g}$ iff $\exists\, T \geq 0$ and for all $w \in X$,

$$|\langle w, PS(T)\binom{f}{g}\rangle_X| \leq (\rho - \sigma)\left(\int_0^T \|B_p^* S^*(T-\tau)P^* w\|_X^2 \, d\tau\right)^{\frac{1}{2}} \qquad (5.13)$$

holds. Let us compute the following terms:

$$\langle w, PS(T)\binom{f}{g}\rangle_X = \langle \sum_{n=1}^{\infty} \hat{w}(n)\sin n\zeta \,,\, \sum_{k=1}^{\infty} \sin k\zeta[\frac{1}{kc}\,\hat{g}(k)\sin kcT + \hat{f}(k)\cos kcT]\rangle_X$$

$$= \frac{\pi}{2}\sum_{n=1}^{\infty} \hat{w}(n)[\frac{1}{nc}\,\hat{g}(n)\sin ncT + \hat{f}(n)\cos ncT] \ ;$$

while

$$\int_0^T \|B_p^* S^*(T-\tau)P^* w\|_X^2 \, d\tau$$

$$= \int_0^T \|\sum_{n=1}^{\infty} \sin n\zeta[\frac{1}{nc}\,\hat{w}(n)\sin nc\,(T-\tau)]\|_X^2 \, d\tau$$

$$= \int_0^T \sum_{n=1}^{\infty} \frac{\pi}{2}\,\frac{\hat{w}^2(n)}{n^2c^2}\,\sin^2 nc\,(T-\tau)\,d\tau$$

$$= \frac{\pi}{4c^2}\sum_{n=1}^{\infty}\frac{\hat{w}^2(n)}{n^2}\left(T - \frac{\sin 2ncT}{2nc}\right) .$$

Observations:

(i) Let $cT = k\pi$, $k \in Z^+$. Then, if $\sum_{n=1}^{\infty} n^2\hat{f}^2(n) \leq K$, for some $K > 0$, then

$$|\langle w, PS(T)\binom{f}{g}\rangle_X| = \frac{\pi}{2}\,|\sum_{n=1}^{\infty}\hat{w}(n)\hat{f}(n)\cos ncT|$$

$$\leq \frac{\pi}{2}\left(\sum_{n=1}^{\infty}\frac{\hat{w}^2(n)}{n^2}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^2\hat{f}^2(n)\right)^{\frac{1}{2}}$$

while the right hand side of (5.13) becomes

$$(\rho - \sigma) \frac{\sqrt{\pi T}}{2c} \left(\sum_{n=1}^{\infty} \frac{\hat{w}^2(n)}{n^2} \right)^{\frac{1}{2}} .$$

Thus, let

$$\frac{\pi}{2}\sqrt{K} \leqslant (\rho - \sigma) \frac{\sqrt{\pi T}}{2}$$

or $\quad T \geqslant \dfrac{\pi K}{(\rho - \sigma)^2}$

and let k be an integer such that $k \geqslant \dfrac{cK}{(\rho - \sigma)^2}$. Then, if $T = \dfrac{k\pi}{c}$, the system (5.9) with (5.10) and (5.11) is null-controllable at $\binom{f}{g}$ on $[0, T]$ for $\sum_{n=1}^{\infty} n^2 f^2(n)$ exists.

(ii) If $\hat{g}(n) = 0$ and $\hat{f}(n) = \dfrac{1}{n^\alpha}$, $\frac{1}{2} < \alpha < 1$, then $\sum_{n=1}^{\infty} \hat{f}^2(n) < \infty$. Consider $\{w_k\} \subset L_2[0, \pi]$ such that $\hat{w}_k(n) = \delta_{kn}$. Then,

$$|\langle w_k, PS(T)\binom{f}{0}\rangle_X| = \frac{\pi}{2}\hat{f}(k)\cos kcT = \frac{\pi}{2k^\alpha}\cos kcT$$

and the left hand side of (5.13) with $w = w_k$ becomes

$$(\rho - \sigma) \frac{\sqrt{\pi}}{2ck} \left(T - \frac{\sin 2kcT}{2kc} \right)^{\frac{1}{2}}$$

$$\leqslant \frac{\sqrt{\pi}}{2ck} (\rho - \sigma)\left(T + \frac{1}{2c}\right)^{\frac{1}{2}} .$$

We observe that for each $T \geqslant 0$, there are infinitely many $k \to \infty$ such that $\cos kcT \geqslant \frac{1}{2}$ and

$$k^{(1-\alpha)} > \frac{2(\rho - \sigma)}{c\sqrt{\pi}} \left(T + \frac{1}{2c}\right)^{\frac{1}{2}} . \qquad (5.14)$$

Hence, for each $T \geqslant 0$, $\exists\, w_k$ as above such that (5.13) cannot be satisfied with $f \in L_2[0, \pi]$, $\hat{f}(n) = \dfrac{1}{n^\alpha}$, $\frac{1}{2} < \alpha < 1$. Therefore, the system (5.9) with (5.10) and (5.11) is not locally null-controllable at the origin.

6. Duality

In this section, we develop a duality theory for controllability and observability similar to the finite dimensional case. Consider the control system

$$\frac{dx}{dt} + A(t)x = \sum_{i=1}^{\mu} B_i(t)u_i(t)$$
$$y_j(t) = C_j(t)x(t) \quad , \qquad j = 1, \ldots, \mu \,. \tag{6.1}$$

The assumptions on $A(t)$, $B_i(t)$, $u_i(t)$, $i = 1, \ldots, \mu$, are as before. In addition, we assume for each $t \geq 0$, $C_j(t)$, $j = 1, \ldots, \mu$, are bounded operators from X to Y_j and are strongly continuous in $t \geq 0$. X, U_i and Y_i, $i = 1, \ldots, \mu$, are Hilbert spaces. Furthermore, if the evolution operator $S(t, \tau) \in B(X)$ can be defined on all $t, \tau \in \mathbb{R}^+$ and strongly continuous in $t, \tau \in \mathbb{R}^+$ such that

(a) $S(t, \tau)\, S(\tau, s) = S(t, s)$, $\quad \forall\, t, \tau, s \in \mathbb{R}^+$

$S(\tau, \tau) = I$, $\quad \tau \in \mathbb{R}^+$;

(b) $\frac{\partial}{\partial t} S(t, \tau)$ and $\frac{\partial}{\partial t} S(\tau, t)$ exist in the strong topology and belong to $B(X)$ for $t, \tau \in \mathbb{R}^+$ and it is also strongly continuous in t ;

(c) $R(S(t, \tau)) \subseteq D(A)$ and $R(S^*(\tau, t)) \subseteq D(A^*)$, $t, \tau \in \mathbb{R}^+$;

then

$$\frac{d}{dt} S^*(\tau, t) = A^*(t) S^*(\tau, t) \,, \qquad t, \tau \in \mathbb{R}^+ \,. \tag{6.2}$$

Thus, $S^*(\tau, t)$ is called the dual evolution operator. Now, we define the formal adjoint system as follows

$$\frac{dz}{dt} - A^*(t)z = \sum_{i=1}^{\mu} C_i^*(t)v_i(t)$$

$$w_j(t) = B_j^*(t)z(t) \tag{6.3}$$

where the state space $Z = X$, output spaces $W_i = U_i$ and input spaces $V_i = Y_i$, $i = 1, \ldots, \mu$. Then, Z, W_i and V_i are Hilbert spaces. The constraints for v_i are

$$L_2^{\sigma_i}(0, T; V_i) \quad ,$$

$$L_2^{\sigma_i}(V_i) = \bigcup_{T \geq 0} L_2^{\sigma_i}(0, T; V_i) ,$$

$\sigma_i > 0$, $i = 1, \ldots, \mu$. We restrict $p_i = q_i = 2$, $i = 1, \ldots, \mu$. In view of (6.2), we see that the mild solution for the system (6.3) is

$$z(t) = S^*(0, t)z_o + \sum_{i=1}^{\mu} \int_o^t S^*(\tau, t)C_i^*(\tau)v_i(\tau)d\tau . \tag{6.4}$$

Remark 6.1. (i) As in Example 5.3, the infinitesimal generator $-\hat{A}$ in the transformed wave equation (5.12) generates a strongly continuous group $\{S(t)\}$, $t \in \mathbb{R}$, which is a good example of the existence of $S(t, \tau) = S(t - \tau)$ described above.

(ii) If A is a bounded operator on X, then $S(t) = \exp(tA)$, $t \in \mathbb{R}$, is a strongly continuous group.

Theorem 6.1. The adjoint system (6.3) is locally null-controllable at the origin on $[0, T]$ iff $\exists\, \varepsilon > 0$ and for all $h \in Z^* = X$,

$$\varepsilon \| S(0, T)h \|_X \leq \sum_{i=1}^{\mu} \sigma_i \left(\int_o^T \| C_i(\tau)S(\tau, T)h \|_{Y_i}^2 d\tau \right)^{\frac{1}{2}} \tag{6.5}$$

holds or equivalently,

$$\sum_{i=1}^{\mu} \int_0^T ||C_i(\tau)S(\tau, T)h||_{Y_i}^2 \, d\tau \geq \gamma ||S(0, T)h||_X^2 \tag{6.6}$$

for some $\gamma > 0$.

Proof: In view of (6.4) and theorem 3.2 with $x_1 = x_o = 0$, we have (6.5). If (6.5) holds, then $\exists\, k \in \{1, \ldots, \mu\}$ such that

$$\int_0^T ||C_k(\tau)S(\tau, T)h||_{Y_k}^2 \, d\tau \geq \frac{\varepsilon^2}{\mu^2 \sigma_k^2} ||S(0, T)h||_X^2 .$$

Let

$$\sigma_{max} = \max_{i=1,\ldots,\mu} \{\sigma_i\} , \qquad \sigma_{min} = \min_{i=1,\ldots,\mu} \{\sigma_i\} .$$

If we take $\gamma = \dfrac{\varepsilon^2}{\mu^2 \sigma_{max}^2}$, then (6.6) holds. Conversely, if (6.6) holds for all $h \in X$, then $\exists\, j \in \{1, \ldots, \mu\}$ such that

$$\int_0^T ||C_j(\tau)S(\tau, T)h||_{Y_j}^2 \, d\tau \geq \frac{\gamma}{\mu} ||S(0, T)h||_X^2 .$$

If we take $\varepsilon = \left(\frac{\gamma}{\mu}\right)^{\frac{1}{2}} \sigma_{min}$, then (6.5) holds for all $h \in X$.

Definition 6.1. The system (6.1) is said to be continuously observable on $[0, T]$ if the initial state can be determined by the inputs and outputs on $[0, T]$ and the observability operator

$$M(0, T)x_o = \sum_{i=1}^{\mu} \int_0^T S^*(\tau, 0)C_i^*(\tau)C_i(\tau)S(\tau, 0)x_o \tag{6.7}$$

has bounded inverse.

Since input constraints will not affect observability, we have the following well known result [11] similar to the finite dimensional case.

Theorem 6.2. The system (6.1) is continuously observable on $[0, T]$ iff the observability operator $M(0, T)$ is positive definite in the sense that $\exists\, \gamma > 0$ such that

$$<M(0, T)x_o, x_o>_X \; > \; \gamma \|x_o\|^2 \; .$$

Since

$$\sum_{i=1}^{\mu} \int_o^T \|C_i(\tau)\, S(\tau, T)h\|^2_{Y_i} \, d\tau$$

$$= \sum_{i=1}^{\mu} \int_o^T <C_i(\tau)\, S(\tau, T)h, \;\; C_i(\tau)\, S(\tau, T)h>_{Y_i} \, d\tau$$

$$= \sum_{i=1}^{\mu} \int_o^T <S^*(\tau, T)\; C_i^*(T) C_i(\tau)\, S(\tau, T)h, \; h>_{Y_i} \, d\tau$$

$$= \;\; <S^*(0, T)M(0, T)S(0, T)h, \; h>_X = <M(0, T)S(0, T)h, \; S(0, T)h> \;\; ,$$

then (6.6) holds iff $M(0, T)S(0, T)$ is positive definite.

Define the controllability operator

$$W(0, T)x_o = \sum_{i=1}^{\mu} \int_o^T S(0, \tau)B_i(\tau)\, B_i^*(\tau)\, S^*(0, \tau)x_o \tag{6.8}$$

for system (6.1). Then, in view of theorem 3.2 and the above computation, the system (6.1) is locally null-controllable at the origin on $[0, T]$ iff $W(0, T)S^*(T, 0)$ is positive definite, where $W(0, T)$ is the observability operator for the system (6.3). Hence, we have proved the following duality theorem.

Theorem 6.3. The system (6.1) is locally null-controllable at the origin on $[0, T]$ iff the adjoint system (6.3) is continuously observable on $[0, T]$; the system (6.1) is continuously observable on $[0, T]$ iff the adjoint system (6.3) is locally null-controllable at the origin on $[0, T]$.

Now, we would like to characterize the constraints of the controls in the adjoint system. Define

$$\Sigma_\varepsilon = \{\sigma = (\sigma_1, \ldots, \sigma_\mu) : \sigma_i \geq 0 \text{ such that (6.5) holds for all } h \in X\}$$

and

$$\|\sigma\|_2 = \left(\sum_{i=1}^{\mu} \sigma_i^2\right)^{\frac{1}{2}} .$$

Then, Σ_ε is closed for each $\varepsilon > 0$.

Lemma 6.1. If $\Sigma_\varepsilon \neq \phi$, $\varepsilon > 0$, then $\exists\, \hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_\mu) \in \Sigma_\varepsilon$ such that $\hat{\sigma}$ is of minimal norm.

Proof: Since Σ_ε is closed and bounded below and (6.5) is continuous in σ , thus, $\hat{\sigma}$ exists and belongs to Σ_ε .

Let

$$E_\sigma = \{\varepsilon > 0 : \text{(6.5) holds for all } h \in X\} .$$

The following lemma is easy to prove.

Lemma 6.2. If $E_\sigma \neq \phi$, $\sigma_i \geq 0$, then $\hat{\varepsilon} \in E_\sigma$ and assumes the maximal value.

Remark 6.2. The duality theorem (6.3) still holds if X, Y_i, U_i , $i = 1, \ldots, \mu$ are reflexive Banach spaces with $Z = X^*$, $W_i = U_i^*$, $V_i = Y_i^*$. However, (6.7) and (6.8) cannot be defined since an inner product may not exist. In addition, the necessary and sufficient condition for the system (6.1) to be continuously observable on $[0, T]$ is that $\exists \gamma > 0$,

$$\sum_{i=1}^{\mu} \int_0^T \|C_i(\tau)\, S(\tau, 0)h\|_{Y_i}^2 \, d\tau > \gamma \|h\|_X^2 \; .$$

holds for all $h \in X$.

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